An output feedback dynamic control scheme is proposed for robust reference position trajectory tracking tasks in a two degree-of-freedom linear mass-spring-damper vibrating mechanical system. A simplified mathematical model of the differentially flat mechanical system is employed for control design purposes. Robustness is considered against parametric uncertainty and input un-modeled dynamics. Disturbances are locally modeled by Taylor polynomial expansions and directly compensated by the controller. The proposed control scheme only requires measurements of the system position output variable. Some computer simulation results are provided to show the robust and efficient dynamic performance of the controller presented in this work.

1. Introduction

Output feedback control of under-actuated vibrating mechanical systems subjected to endogenous and exogenous disturbances represents a long standing challenging research topic in active vibration control. In previous works, it has been shown theoretically and experimentally the potential application of the Generalized Proportional-Integral (GPI) control design methodology for robust perturbation rejection in a mass-spring-damper mechanical system of one degree of freedom [1] and in the synthesis of active vibration absorption schemes to attenuate undesirable harmonic vibrations [2, 3].

Generalized proportional-integral controllers and integral reconstructors of the state variables for constant linear systems were introduced by Fliess et al. in [4] to avoid the use of asymptotic state observers. This control approach is based on the module-theoretic framework for linear systems and Mikusiński operational calculus. Thus, the GPI control represents a good alternative for active control of controllable mechanical systems using only measurements of the output variables.

In this paper, an output feedback dynamic control scheme is proposed for robust reference position trajectory tracking tasks in a two degree-of-freedom under-actuated linear mass-spring-damper vibrating mechanical system perturbed by an unknown secondary linear mass-spring-damper mechanical system. The presented control approach can also be extended for completely actuated or under-actuated multi-degree-of-freedom mechanical systems. A simplified mathematical model of the transformed differentially flat mechanical system, involving the known system dynamics solely,
is employed for control design purposes. The un-modeled dynamics and small parametric uncertainty are lumped into a state dependent perturbation affecting the known system dynamics. The perturbation signal is locally modeled by a family of Taylor polynomials of forth degree to get a better approximation of this disturbing signal into a small time window. Then, the GPI control, differential flatness [5] and the Taylor polynomial expansion of the perturbation signal are properly combined for the synthesis of the presented control scheme, which only requires measurements of the system position output variable to be controlled. Some preliminary computer simulation results are provided to show the robust and efficient dynamic performance of the controller for the tracking of a reference trajectory described by a Bézier interpolation polynomial. The motion planning is specified to take the output variable from the rest position to a desired nominal position.

2. Mass-spring-damper system

Consider the mass-spring-damper mechanical system shown in Fig. 1. The generalized coordinates are the positions of the mass carriages, \( x_i, i = 1, 2, \ldots, n \). In addition, \( u \) represents the force control input, \( y = x_2 \) is the position output variable to be controlled, and \( m_i, k_i \) and \( c_i \) denote mass, stiffness and viscous damping associated to the \( i \)-th degree-of-freedom.

The mathematical model of this mechanical system is described by the coupled ordinary differential equations

\[
\begin{align*}
    m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 &= k_2 (x_2 - x_1) + u \\
    m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 (x_2 - x_1) &= k_3 (x_3 - x_2) \\
    \vdots \\
    m_{n-1} \ddot{x}_{n-1} + c_{n-1} \dot{x}_{n-1} + k_{n-1} (x_{n-1} - x_{n-2}) &= k_n (x_n - x_{n-1}) \\
    m_n \ddot{x}_n + c_n \dot{x}_n + k_n (x_n - x_{n-1}) &= 0 \\
    y &= x_2
\end{align*}
\]

In this paper, it is considered that the dynamics associated with the first two masses is completely known. Therefore, the two degree-of-freedom under-actuated mechanical system is perturbed by the unknown state dependent spring force \( f_{k_3} = k_3 (x_3 - x_2) \).

Defining the state variables as \( z_1 = x_1, z_2 = \dot{x}_1, z_3 = x_2 \) and \( z_4 = \dot{x}_2 \), one obtains the state space description

\[
\begin{align*}
    \dot{z}_1 &= z_2 \\
    \dot{z}_2 &= -\frac{k_1 + k_2}{m_1} z_1 - \frac{c_1}{m_1} z_2 + \frac{k_2}{m_1} z_3 + \frac{1}{m_1} u \\
    \dot{z}_3 &= z_4 \\
    \dot{z}_4 &= \frac{k_1}{m_2} z_1 - \frac{k_2}{m_2} z_3 - \frac{c_2}{m_2} z_4 + \frac{1}{m_2} \dot{f}_{k_3} \\
    y &= z_3
\end{align*}
\]
One can easily verify that the mechanical system (2) is completely controllable from the control input variable \( u \). Therefore, the mass-spring-damper system exhibits the differential flatness property. Hence all state variables and the control input can be parameterized in terms of the flat output \( y \) and a finite number of its time derivatives [5].

Then from \( y \) and its time derivatives up to fourth order, the unperturbed differential parameterization results as

\[
\begin{align*}
z_1 &= \frac{k_2}{k_1} y + \frac{c_2}{k_1} \ddot{y} + \frac{m_2}{k_1} \dot{y} \\
z_2 &= \frac{k_2}{k_1} y + \frac{c_2}{k_1} \ddot{y} + \frac{m_2}{k_1} \dot{y}^{(3)} \\
z_3 &= y \\
z_4 &= \dot{y} \\
u &= \frac{k_2}{k_1} y + \left( \frac{c_2}{k_1} \ddot{y} + \frac{c_1 k_2}{k_1} + \frac{c_2 k_2}{k_1} \right) \dot{y} \\
&\quad+ \left( \frac{m_2}{k_1} + \frac{k_2 m_1}{k_1} + \frac{k_2 m_2}{k_1} \right) \ddot{y} \\
&\quad+ \left( \frac{c_1 m_2}{k_1} + \frac{c_2 m_1}{k_1} \right) y^{(3)} + \frac{m_1 m_2}{k_1} y^{(4)}
\end{align*}
\]  

Thence, the flat output \( y \) satisfies the perturbed input-output ordinary differential equation

\[
y^{(4)} + a_3 y^{(3)} + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b u + \varphi(t)
\]

with

\[
\begin{align*}
a_0 &= \frac{k_2^2}{m_1 m_2} \\
a_1 &= \frac{c_1 k_2}{m_1 m_2} + \frac{c_2 k_1}{m_1 m_2} + \frac{c_2 k_2}{m_1 m_2} \\
a_2 &= \frac{k_1}{m_1} + \frac{k_2}{m_2} + \frac{c_1 c_2}{m_1 m_2} \\
a_3 &= \frac{c_1}{m_1} + \frac{c_2}{m_2} \\
b &= \frac{k_1}{m_1 m_2}
\end{align*}
\]

In the transformed differentially flat mechanical system dynamics (4), the term \( \varphi(t) \) has been included to consider the influence of the disturbance force \( f_{k_3} \) as well as small parametric uncertainties. Then, from (4) one can get the following differential flatness based controller to asymptotically track some reference position trajectory \( y^*(t) \):

\[
u = \frac{1}{b} \left( v + a_0 y + a_1 \dot{y} + a_2 \ddot{y} + a_3 y^{(3)} - \varphi \right)
\]

with

\[
v = y^{(4)*} - a_3 \left( y^{(3)} - y^{(3)*} \right) - a_2 \left( \ddot{y} - \ddot{y}^* \right) - a_1 \left( \dot{y} - \dot{y}^* \right) - a_0 \left( y - y^* \right)
\]

However, the controller (5) requires measurements of the position, velocity, acceleration and jerk signals as well as the perfect knowledge of the disturbance \( \varphi(t) \). Thus, in the next section, the GPI control design methodology, differential flatness and Taylor polynomial expansions of the disturbance signal are advantageously employed for the synthesis of a robust output feedback control scheme which only needs measurements of the position output variable.
3. Output feedback Control with integral compensation

For control design purposes, it is assumed that the disturbance signal \( \varphi(t) \) can be locally approximated by the family of Taylor polynomials of fourth degree [1]

\[
\varphi(t) \approx \sum_{i=0}^{4} p_i t^i
\]  

(7)

where all the coefficients \( p_i \) are completely unknown.

Then, the local dynamics of the mechanical system can be described as

\[
y^{(4)} + a_3 y^{(3)} + a_2 \dot{y} + a_1 \ddot{y} + a_0 y = bu + \sum_{i=0}^{4} p_i t^i
\]  

(8)

From this equation, one can obtain by successive integrations the following integral reconstruc-
tors for the time derivatives up to third order of the flat output \( y \):

\[
\hat{\dot{y}} = -a_0 \int y^{(3)} - a_1 \int y^{(2)} - a_2 \int y - a_3 y + b \int u
\]

\[
\hat{\ddot{y}} = -a_0 \int y^{(2)} - a_1 \int y - a_2 \int \ddot{y} - a_3 \dot{y} + b \int u
\]

\[
\hat{y}^{(3)} = -a_0 \int y - a_1 y - a_2 \dot{y} - a_3 \ddot{y} + b \int u
\]  

(9)

where the integral \( \int_{t_0}^{t} \eta(\tau) d\tau \) is described by \( \int \eta \) and \( \int_{t_0}^{t} \int_{\tau_0}^{\tau} \eta(\rho) d\rho d\tau \) by \( \int^{(2)} \eta \), and so on.

Note that the system initial conditions and the coefficients \( p_i \) were not intentionally taken into account in the integral reconstruction of the time derivatives of the position output signal. Thus, the structural estimates (9) differ from the actual values by an algebraic polynomial up to seventh degree:

\[
\xi(t) = \sum_{i=0}^{7} \lambda_i t^i
\]  

(10)

where the constants \( \lambda_i \) depend on the unknown initial conditions and the coefficients of the distur-
bance model (7).

Then, the following dynamic controller using integral reconstruction of the time derivatives is
proposed for reference position trajectory tracking tasks:

\[
u = \frac{1}{b} \left( \dot{y} + a_0 y + a_1 \dot{y} + a_2 \ddot{y} + a_3 \dddot{y} \right)
\]  

(11)

with

\[
v = y^{(4)} - \alpha_{11} \left( \ddot{y} - \ddot{y}^* \right) - \alpha_{10} \left( \dot{y} - \dot{y}^* \right)
\]

\[
-\alpha_9 \left( \ddot{y} - \ddot{y}^* \right) - \alpha_8 \left( y - y^* \right) - \sum_{k=0}^{7} \alpha_k \int_{t_0}^{t} \left( y - y^* \right)
\]  

(12)

The last integral terms yields error compensation, eliminating those destabilizing effects of the
structural estimation errors.
Substitution of the controller (11) into (8), and differentiating eight times the resulting expression with respect to time, lead to the closed-loop tracking error dynamics, $e = y - y^*$:

$$e^{(12)} + \sum_{i=0}^{11} \alpha_i e^{(i)} = 0 \quad (13)$$

Then by selecting the design parameters $\alpha_i$ so that the characteristic polynomial associated with the tracking error dynamics (13) is a Hurwitz polynomial results in a globally exponentially asymptotically stable equilibrium point, $e = 0$.

4. Simulation results

The performance of the proposed control scheme was verified for a mechanical system with three degrees of freedom characterized by the set of parameters given in Table 1.

<table>
<thead>
<tr>
<th>$m_1$</th>
<th>2 kg</th>
<th>$m_2$</th>
<th>2 kg</th>
<th>$m_3$</th>
<th>3 kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>2 Ns/m</td>
<td>$c_2$</td>
<td>1 Ns/m</td>
<td>$c_3$</td>
<td>1 Ns/m</td>
</tr>
<tr>
<td>$k_1$</td>
<td>400 N/m</td>
<td>$k_2$</td>
<td>400 N/m</td>
<td>$k_3$</td>
<td>400 N/m</td>
</tr>
</tbody>
</table>

Fig. 2 depicts the reference trajectory $y^*(t)$ specified for the position output variable, $y = x_2$. The motion profile $y^*(t)$ planned for the mechanical system is described as

$$y^* = \begin{cases} \bar{y}_1 + (\bar{y}_2 - \bar{y}_1) \psi(t, T_1, T_2) & \text{for } 0 \leq t < T_1 \\ \bar{y}_2 & \text{for } T_1 \leq t \leq T_2 \\ \bar{y}_2 & \text{for } t > T_2 \end{cases}$$

Where $\bar{y}_1 = 0$ m, $\bar{y}_2 = 0.01$ m, $T_1 = 0$ s, $T_2 = 5$ s, and $\psi(t, T_1, T_2)$ is a Bézier polynomial, with $\psi(T_1, T_1, T_2) = 0$ and $\psi(T_2, T_1, T_2) = 1$, given by

$$\psi(t) = \left( \frac{t - T_1}{T_2 - T_1} \right)^5 [r_1 - r_2 \left( \frac{t - T_1}{T_2 - T_1} \right)$$

$$+ r_3 \left( \frac{t - T_1}{T_2 - T_1} \right)^2 - ... - r_6 \left( \frac{t - T_1}{T_2 - T_1} \right)^5]$$

With $r_1 = 252, r_2 = 1050, r_3 = 1800, r_4 = 1575, r_5 = 700, r_6 = 126$.

The design parameters for the controller were selected to have a twelfth order closed-loop characteristic polynomial of the form

$$P_c(s) = \left( s^2 + 2\zeta \omega_n s + \omega_n^2 \right)^6$$

With natural frequency $\omega_n = 100$ rad/s and damping ratio $\zeta = 0.7071$.

Figs. 3-5 display the closed-loop responses of the two degrees-of-freedom mechanical system using the proposed output feedback control scheme (11). Here, the system is perturbed by the unmodeled dynamics associated with the unknown secondary mechanical system $(m_3, c_3, k_3)$. For this preliminary operation scenario, one can observe the satisfactory tracking of the reference position trajectory.

Therefore, the presented design methodology represents an alternative for the controller synthesis for under-actuated perturbed linear mass-spring-damper mechanical systems of two degrees of freedom employing only measurements of the position output variable. Evidently, if other state variables were available the controller design would be greatly simplified. Moreover, some disturbances are also admitted such as small parametric uncertainty and un-modeled dynamics of the same structure of the mass-spring-damper system to be controlled. Nevertheless, the control approach can be extended for multi-degree-of-freedom mechanical systems perturbed by this kind of disturbance forces.
5. Conclusions

In this paper, an output feedback dynamic control scheme has been proposed for robust reference position trajectory tracking tasks for a topology of under-actuated linear mass-spring-damper mechanical systems of two degrees of freedom subjected to disturbances due to un-modeled dynamics and possibly small parametric uncertainties. The state dependent disturbances are induced by couplings of the system with another unknown mass-spring-damper system. Thus, a simplified mathematical model of the perturbed system dynamics was employed in the control design process. Moreover, the perturbation signal was locally approximated by a family of Taylor polynomials of forth degree in order to reduce the complexity of the controller design. Then, the GPI control, differential flatness and the Taylor polynomial expansion of the perturbation signal were properly combined for the synthesis of the presented control scheme. An important feature of the control scheme is its capability of rejecting disturbances using only measurements of the position output variable and simultaneously to perform the motion planning specified for the mechanical system. The prelimi-

Figure 2. Position reference trajectory planned for the mechanical system.

Figure 3. Closed-loop response of the under-actuated position output variable $y$. 
nary computer simulation results show a satisfactory performance of the control scheme. Therefore, the presented control approach represents a good alternative for the controller synthesis for under-actuated perturbed linear mass-spring-damper systems of two degrees of freedom. In addition, the presented methodology can be extended for active vibration control of completely actuated or under-actuated multi-degree-of-freedom mechanical systems.

REFERENCES


