A Note on the Influence of Intermediate Restraints and Hinges in Frequencies and Mode Shapes of Beams

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This note deals with the free transverse vibration of a beam with two arbitrarily located internal hinges, four intermediate elastic restraints, and ends elastically restrained against rotation and translation. The method of separation of variables is used for the determination of the exact frequencies and mode shapes. New results are presented for different boundary conditions and restraint conditions in the internal hinges.

The mathematical model is also used to study the influence on the frequencies and mode shapes of varying intermediate supports that are located at the nodal points of higher modes. A detailed numerical study on the effects of the locations of intermediate translational restraints and their stiffness on the natural frequencies and mode shapes is performed for different boundary conditions. The effect of the presence of the internal hinges is also analysed. Graphs and tables of the non-dimensional frequencies and the corresponding mode shapes are given in order to illustrate the behaviour of frequency parameters and the presence of mode shape switching.

1. INTRODUCTION

There has been extensive research into the vibration of Euler–Bernoulli beams with elastic restraints. It is not possible to give a reasonable and detailed account of this great amount of information; nevertheless, some relevant references will be cited. Particularly, several investigators have studied the influence of elastic restraints at the ends of vibrating beams.[1–16] Exact frequency and normal mode shape expressions have been derived for uniform beams with elastically restrained ends against rotation and translation.[17] Excellent handbooks have appeared in the literature giving frequencies, tables and mode shape expressions.[18,19]

The problem of the vibrations of beams that are elastically restrained at intermediate points has also been extensively treated. One of the earliest works was performed by Lee and Saibel who analysed the problem of free vibrations of a constrained beam with intermediate elastic supports.[20] Rutemberg presented eigenfrequencies for a uniform cantilever beam with a rotational restraint at an intermediate position.[21] Lau extended Rutemberg's results with an additional translational restriction.[22] Maurizi and Bambill analysed the transverse vibrations of clamped beams with an intermediate translational restraint.[23] Rao analysed the frequencies of a clamped-clamped uniform beam with intermediate elastic support.[24] De Rosa et al. studied the free vibrations of stepped beams with intermediate elastic supports.[25] Ewing and Mirsafian analysed the forced vibrations of two beams joined with a non-linear rotational joint.[26] Arenas and Grossi presented exact and approximate frequencies of a uniform beam, with one end spring-hinged and a rotational restraint in a variable position.[27] Grossi and Albarracín determined the exact eigenfrequencies of a uniform beam with intermediate elastic constraints.[28]

The minimum stiffness of an elastic traslational restriction that raises a natural frequency of a beam to its upper limit has been investigated by several researchers. Courant and Hilbert have demonstrated that the optimum location of a rigid support should be at the nodal points of a higher vibration mode.[29] Akesson and Olhoff showed that in the case of elastic supports the optimum locations are the same as that of rigid supports, and there exists a minimum stiffness of an additional elastic support whenever the fundamental frequency of a uniform cantilever beam is increased to its maximum.[30] Wang determined the minimum stiffness of an internal elastic support to maximize the fundamental frequency of a vibrating beam.[31] Wang et al. derived the closed-form solution for the minimum stiffness of a simple point support that raises a natural frequency of a beam to its upper limit.[32] Albarracín et al. detected a rather curious situation of changes in frequency values and mode shapes when an intermediate translational restraint is placed in a beam that is simply supported at both ends.[33]

There is only a limited amount of information for the vibration of beams with internal hinges. Wang and Wang studied the fundamental frequency of a beam with an internal hinge with an axial force.[34] Chang et al. investigated the dynamic response of a beam with an internal hinge, subjected to a random moving oscillator.[35] Grossi and Quintana investigated the natural frequencies and mode shapes of a non homogeneous tapered beam subjected to general axial forces, with an arbitrarily located internal hinge and elastic supports and ends that are elastically restrained against rotation and translation.[36]

The above review of the literature reveals that many efforts have been devoted to the analysis of the influence of elastic restraints parameters, located at the ends and intermediate points, on the dynamics characteristics of beams. However, the influence on the frequencies and mode shapes of varying intermediate supports located at the nodal points of higher modes has been studied only for classical end conditions. There is no paper that presents a complete analysis of the mentioned effects of intermediate elastic supports in a beam generally restrained at both ends. Also, in this subject the presence of internal hinges has not been treated.

The aim of the present note is to investigate the natural frequencies and mode shapes of a beam with two arbitrarily located internal hinges, four intermediate elastic restraints and ends elastically restrained against rotation and translation. Adopting the adequate values of the rotational and translational restraints parameters at the ends, all the possible combinations of classical end conditions, (i.e., clamped, simply supported, sliding and free) can be generated. The presence of the two hinges and the intermediate elastic restraints in particular, allows for the inclusion of a hinge located at an intermediate point and a translational restraint located at a different point. This property will prove to be valuable in studying the influence of a translational restraint located at a node of a higher mode of vibration. The existence of a critical value of the dimensionless restraint parameter which determines the interchange of roles of the corresponding modal shapes of two consecutive non-dimensional frequency parameters is demonstrated. More specifically, whenever there is no internal hinge and the beam is simply supported, it is demonstrated that some frequencies increase as the stiffness of the intermediate support increases; if this parameter assumes a critical value, the symmetric modes shift to anti-symmetric modes and vice versa. The existence of an analogue phenomenon in the case of other boundary conditions is established; this also exits when there is an internal hinge.

The classical method of the separation of variables has been used for the determination of the exact frequencies and mode shapes. The algorithm developed can be applied to a wide range of elastic restraint conditions. The effects of the variations of the elastic restraints on the switching of the mode shape order and the influence of the internal hinges are investigated. Tables and figures are given for frequencies, and two-dimensional plots for mode shapes are included. A great number of problems were solved and, since the number of cases is prohibitively large, results are presented for only a few cases. The present note is organized first by the brief history stated above. In Section 2, a rigorous treatment of techniques of the calculus of variations to obtain the governing differential equations, the boundary conditions and the transitions conditions is presented. In Section 3, the method of the separation of variables is used for the determination of the exact frequencies and mode shapes. In Section 4, the influence of intermediate translational restraints is considered, and the analyses of the most important cases are included. Finally, Section 5 contains the conclusions of this note.

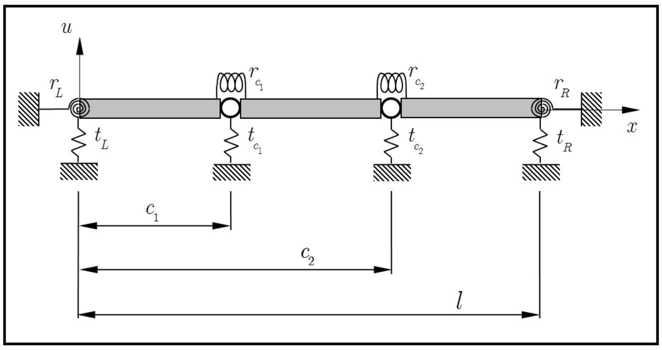


Figure 1. Mechanical system under study.

2. THE BOUNDARY VALUE PROBLEM

Let us consider a beam of length l, which has elastically restrained ends, is constrained at two intermediate points and has two internal hinges, as shown in Fig. (1). The beam system is made up of three different spans, which correspond to the intervals $[0, c_1]$, $[c_1, c_2]$ and $[c_2, l]$ respectively. It is assumed that the ends and the internal hinges are elastically restrained against translation and rotation. The rotational restraints are characterised by the parameters $r_L, r_R, r_{c_i}, i = 1{,}2$ and the translational are characterized by $t_L, t_R, t_{c_i} = 1{,}2$. Adopting the adequate values of the parameters r_L, r_R and t_L, t_R, all the possible combinations of classical end conditions can be generated. By using $t_c, r_{c_i}, i = 1{,}2$, the effects of the internal hinges and intermediate restraints are taken into account.

In order to analyse the transverse planar displacements of the system under study, we suppose that the vertical position of the beam at any time t is described by the function $u = u(x,t), x \in [0,l]$. It is well known that at time t, the kinetic energy of the beam can be expressed as

$$T_b = \frac{1}{2}\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i} (\rho A)_i(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2 dx; \tag{1}$$

where $(\rho A)_i = \rho_i A_i$ denotes the mass per unit length of the $i - th$ span and $c_0 = 0, c_3 = l$.

The total potential energy due to the elastic deformation of the beam, the elastic restraints at the ends and the intermediate elastic restraints, is given by:

$$U = \frac{1}{2}\left\{\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i} (EI)_i(x)\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)^2 dx + \sum_{i=0}^{3}\left(r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t) - \frac{\partial u}{\partial x}(c_i^-,t)\right)^2 + t_{c_i}u^2(c_i,t)\right)\right\}; \tag{2}$$

where $(EI)_i = E_i I_i$ denotes the flexural rigidity of the $i - th$ span, $r_{c_0} = r_L$, $t_{c_0} = t_L$, $r_{c_3} = r_R$, and $t_{c_3} = t_R$. The notations 0^+, c_i^+, c_i^- and l^- imply the use of lateral limits and lateral derivatives and in consequence in Eq. (2), it is assumed that $\frac{\partial u}{\partial x}(0^-,t) = 0$, $\frac{\partial u}{\partial x}(l^+,t) = 0$.

Hamilton's principle requires that between times t_a and t_b, at which the positions are known, the motion will make station-

$$F(u) = \frac{1}{2}\int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left((\rho A)_i(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - (EI)_i(x)\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)^2\right)dx\right]dt$$
$$-\frac{1}{2}\int_{t_a}^{t_b}\sum_{i=0}^{3}\left[r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t) - \frac{\partial u}{\partial x}(c_i^-,t)\right)^2 + t_{c_i}u^2(c_i,t)\right]dt. \tag{3}$$

$$D = \Big\{u; u(x,\bullet) \in C^2([t_a,t_b]),\ u(\bullet,t) \in C([0,l]),\ u(\bullet,t)|_{[c_{i-1},c_i]} \in C^4([c_{i-1},c_i]),\ i = 1,2,3,$$
$$u(x,t_a), u(x,t_b) \text{ predescribed } \forall x \in [0,l]\Big\}. \tag{5}$$

$$D_a = \Big\{v; v(x,\bullet) \in C^2([t_a,t_b]),\ v(\bullet,t) \in C([0,l]),\ v(\bullet,t)|_{[c_{i-1},c_i]} \in C^4([c_{i-1},c_i]),\ i = 1,2,3,$$
$$v(x,t_a) = v(x,t_b) = 0,\ \forall x \in [0,l]\Big\}. \tag{6}$$

ary the action integral $F(u) = \int_{t_a}^{t_b} L dt$ on the space of admissible functions, where the Lagrangian L is given by $L = T_b - U$. In consequence, the energy functional to be considered is given by Eq. (3).

The stationary condition for the functional given by Eq. (3) requires that

$$\delta F(u;v) = 0, \quad \forall v \in D_a; \tag{4}$$

where $\delta F(u;v)$ is the first variation of F at u in the direction v and D_a is the space of admissible directions at u for the space D of admissible functions. In order to make the mathematical developments required by the application of the techniques of the calculus of variations, we assume that $(\rho A)_i \in C([c_{i-1},c_i])$, $(EI)_i \in C^2([c_{i-1},c_i])$, $i = 1,2,3$. The space D is the set of functions $u(x,\bullet) \in C^2([t_a,t_b])$, $u(\bullet,t) \in C([0,l])$, $u(\bullet,t)|_{[c_{i-1},c_i]} \in C^4([c_{i-1},c_i])$, $i = 1,2,3$.

In view of all these observations and since Hamilton's principle requires that at times t_a and t_b the positions are known, the space D is given by Eq. (5).

The only admissible directions v at $u \in D$ are those for which $u + \varepsilon v \in D$ for a sufficiently small ε and $\delta F(u;v)$ exists. In consequence, and in view of Eq. (5), v is an admissible direction at u for D if, and only if, $v \in D_a$ where D_a is given by Eq. (6).

The definition of the first variation of F at u in the direction v, is given by

$$\delta F(u;v) = \frac{dF(u+\varepsilon v)}{d\varepsilon}|_{\varepsilon=0}. \tag{7}$$

The application of Eq. (7) with the expression of F given by Eq. (3) leads to Eq.(8).

The integration by parts of the procedure and the well-known application of the fundamental lemma of the calculus of variations leads to the following boundary value problem:

$$\frac{\partial^2}{\partial x^2}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) + (\rho A)_i(x)\frac{\partial^2 u}{\partial t^2}(x,t) = 0,$$
$$\forall x \in (c_{i-1},c_i), \quad i = 1,2,3,\ t \geq 0; \tag{9}$$

$$r_L\frac{\partial u}{\partial x}(0^+,t) = (EI)_1 0^+\frac{\partial^2 u}{\partial x^2}(0^+,t); \tag{10}$$

$$t_L u(0^+,t) = -\frac{\partial}{\partial x}\left((EI)_1(0^+)\frac{\partial^2 u}{\partial x^2}(0^+,t)\right); \tag{11}$$

$$w(c_i^-,t) = w(c_i^+,t), \quad i = 1,2; \tag{12}$$

$$r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t) - \frac{\partial u}{\partial x}(c_i^-,t)\right) =$$
$$(EI)_i(c_i^-)\frac{\partial^2 u}{\partial x^2}(c_i^-,t), \quad i = 1,2; \tag{13}$$

$$r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t) - \frac{\partial u}{\partial x}(c_i^-,t)\right) =$$
$$(EI)_{i+1}(c_i^+)\frac{\partial^2 u}{\partial x^2}(c_i^+,t), \quad i = 1,2; \tag{14}$$

$$t_{c_i}u(c_i,t) = \frac{\partial}{\partial x}\left((EI)_i(c_i^-)\frac{\partial^2 u}{\partial x^2}(c_i^-,t)\right) -$$
$$\frac{\partial}{\partial x}\left((EI)_{i+1}(c_i^+)\frac{\partial^2 u}{\partial x^2}(c_i^+,t)\right), \quad i = 1,2; \tag{15}$$

$$r_R\frac{\partial u}{\partial x}(l^-,t) = -(EI)_3(l^-)\frac{\partial^2 u}{\partial x^2}(l^-,t); \tag{16}$$

$$t_R u(l^-,t) = \frac{\partial}{\partial x}\left((EI)_3(l^-)\frac{\partial^3 u}{\partial x^3}(l^-,t)\right) \text{ where } t \geq 0. \tag{17}$$

Different situations can be generated by substituting values and/or limiting values of the restraint parameters r_c and t_c. When we consider $r_{c_i} = \infty$, $t_{c_i} = 0$, $i = 1,2$, there are no internal hinges. Now if we consider $r_{c_i} = 0$, $t_{c_i} = 0$, $i = 1,2$, there are internal hinges located at c_1 and c_2 and the articulations are perfect. Finally if we have $0 < r_{c_i} < \infty$, $0 < t_{c_i} < \infty$, $i = 1,2$, there are internal hinges elastically restrained against rotation and supported by the respective translational restraints.

$$\delta F(u;v)=\int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left((\rho A)_i(x)\frac{\partial u}{\partial t}(x,t)\frac{\partial v}{\partial t}(x,t)-(EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\frac{\partial^2 v}{\partial x^2}(x,t)\right)dx-\right.$$
$$\left.-\sum_{i=0}^{3}\left(r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t)-\frac{\partial u}{\partial x}(c_i^-,t)\right)\left(\frac{\partial v}{\partial x}(c_i^+,t)-\frac{\partial v}{\partial x}(c_i^-,t)\right)+t_{c_i}u(c_i,t)v(c_i,t)\right)\right]dt. \quad (8)$$

It is worth noting that this mathematical model allows the inclusion of a hinge located at a point c_i and a translational restraint located at a different point c_j. As stated above this property will prove to be valuable for studying the influence of a translational restraint located at a node of a higher vibration mode.

3. NATURAL FREQUENCIES AND MODE SHAPES

Using the well-known method of the separation of variables, when the mass per unit length and the flexural rigidity at the spans are constant, we assume as solutions of Eq. (9) the functions given by the series

$$u_i(x,t)=\sum_{n=1}^{\infty}u_{i,n}(x)\cos\omega t, \quad i=1,2,3; \quad (18)$$

where $u_{i,n}$ are the corresponding nth nodes of natural vibration, and ω is the radian frequency. Introducing the change of variable $\overline{x}=x/l$ into Eqs. (9) through (17), the functions $u_{i,n}$ are given by

$$u_{i,n}(\overline{x})=A_i\cosh\lambda\overline{x}+B_i\sinh\lambda\overline{x}+C_i\cos\lambda\overline{x}+ \\ +D_i\sin\lambda\overline{x}, \quad \forall\overline{x}\in[a_i,b_i]; \quad (19)$$

where $\overline{c}_i=c_i/l$, $a_1=0$, $b_1=\overline{c}_1$, $a_2=\overline{c}_1$, $b_2=\overline{c}_2$, $a_3=\overline{c}_2$, $b_3=1$, and

$$\lambda^4=\frac{\rho A}{EI}\omega^2 l^4; \quad (20)$$

where $\rho A=(\rho A)_1=(\rho A)_2=(\rho A)_3$ and $EI=(EI)_1=(EI)_2=(EI)_3$. Substituting Eq. (19) into Eq. (18), and then in the boundary conditions given by Eqs. (10), (11), (16), (17) and transition conditions defined by Eqs. (12) to (15), expressed in the new variable $\overline{x}$, we obtain a set of twelve homogeneous equations in the constants $A_i,\ldots,D_i$. Since the system is homogeneous, in order to obtain a non-trivial solution, the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G(T_L,R_L,T_R,R_R,T_{\overline{c}_i},R_{\overline{c}_i},\lambda,\overline{c}_i)=0,\ i=1,2; \quad (21)$$

where

$$T_L=\frac{t_L l^3}{EI},R_L=\frac{r_L l}{EI},T_R=\frac{t_R l^3}{EI}, \\ R_R=\frac{r_R l}{EI},T_{\overline{c}_i}=\frac{t_{c_i}l^3}{EI},R_{\overline{c}_i}=\frac{r_{c_i}l}{EI}, \quad i=1,2. \quad (22)$$

The values of the frequency parameter $\lambda=\left((\rho A/EI)\omega^2\right)^{\frac{1}{4}}l$, were obtained with the classical bisection method and rounded to eight decimal digits.

In order to describe the corresponding boundary conditions the symbolism SS identifies a simply supported end, C a clamped end, F a free end and ER identifies an elastically restrained end. Since the number of cases which can be analysed by the developed algorithm is prohibitively large, results are presented only for a few cases.

Table 1 and Table 2 depict the first three exact values of the frequency parameter λ of a beam with two internal hinges. Different boundary conditions and values of the parameters $\overline{c}_i, i=1,2$ are considered. The corresponding mode shapes are also included. It is worth noting that in order to avoid zero frequencies and to obtain mode shapes, which clearly show the effect of the hinges, a relative small value of the restraint parameters $T_{\overline{c}_1}$ and $T_{\overline{c}_2}$ has been adopted. Table 1 contains symmetrical boundary conditions, and Table 2 includes non-symmetrical boundary conditions. In the case $ER-ER$, the values $T_L=R_L=1000$, $T_R=R_R=100$ have been adopted. It is worth pointing out that $u(\bullet,t)\in C[0,1]$, i.e. the deflection function is only continuous), but it has corner points that only exist at the hinges locations. This property can be observed in the mode shapes included in Tables 1 and 2.

Table 3 depicts the first six exact values of the frequency parameter λ of a beam with two free internal hinges, different boundary conditions, and different values of the restraint parameters $T_{\overline{c}_1}$ and $T_{\overline{c}_2}$, where $\overline{c}_1=1/3$ and $\overline{c}_2=2/3$. In the case $ER-ER$, the values $T_L=R_L=1000$, $T_R=R_R=100$ have been adopted.

4. THE INFLUENCE OF INTERMEDIATE TRANSLATIONAL RESTRAINTS

As stated in Section 1, in the determination of an additional translational restraint required to maximize a natural frequency, Courant and Hilbert have demonstrated that the optimum location of a support should be at the nodal points of a higher vibration mode, and Akesson and Olhoff have demonstrated the same for elastic supports.[29,30] For this reason in all the described cases in this study, the restraint locations coincide with the nodal points of some higher modes.

First the case of an $SS-SS$ beam without hinges and with one flexible support at the mid-point is considered ($\overline{c}_1=0,\overline{c}_2=\overline{c},\overline{c}=0.5$). Fig. 2 the first two exact values of the frequency parameter λ are plotted against the restraint parameter $T_{\overline{c}}$. It is observed that thecurves have a contact point denoted by P_1, and to this point it corresponds with a value, namely $T^{(1,2)}$ of $T_{\overline{c}}$, such that over it the values of λ_1 cannot be raised

BC	$\overline{c}_1$	$\overline{c}_2$	λ_1		λ_2		λ_3	
$F-F$	0.2	0.6	1.375081		1.643200		9.362790	
	0.25	0.75	1.337032		1.618163		8.120665	
	1/3	2/3	1.414102		1.638014		10.669474	
	0.5	0.75	1.445185		1.681667		8.680980	
$SS-SS$	0.2	0.6	1.342282		1.583975		8.273242	
	0.25	0.75	1.315682		1.565026		7.855269	
	1/3	2/3	1.377333		1.564958		9.424777	
	0.5	0.75	1.390513		1.608228		7.173072	
$C-C$	0.2	0.6	3.772331		6.656591		9.738941	
	0.25	0.75	4.686472		6.031293		8.680536	
	1/3	2/3	4.248089		4.958357		10.740080	
	0.5	0.75	3.324057		6.124616		8.655705	

Table 1. Values λ_1, λ_2 and λ_3 of the frequency coefficient λ and mode shapes of a beam with two internal hinges with different symmetrical boundary conditions and values of the parameters $\overline{c}_i, i = 1, 2$. ($T_{\overline{c}_1} = T_{\overline{c}_2} = 1$).

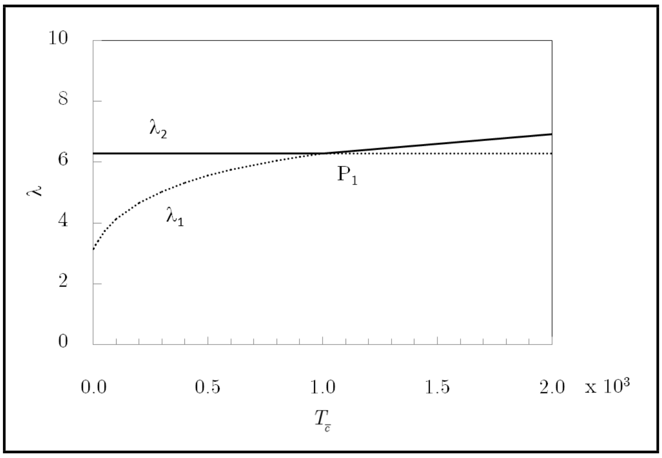


Figure 2. Variation of the first two exact values of the frequency parameter λ as a function of the translational restraint parameter $T_{\overline{c}}$, which corresponds to an $SS-SS$ beam with $\overline{c} = 0.5$.

$T_{\overline{c}}$	λ_1	λ_2
0	3.14159265	6.28318531
$T^{(1,2)} - \varepsilon$	6.28318520	6.28318531
$T^{(1,2)} = 995.9135$	6.28318531	6.28318531
$T^{(1,2)} + \varepsilon$	6.28318531	6.28318541

Table 4. Values λ_1, λ_2 of the frequency coefficient λ and mode shapes of an $SS-SS$ beam with $T^{(1,2)} = 995.9135$ and $\overline{c} = 0.5$.

further whereas the values of the coefficient λ_2 increases. This phenomenon suggests the possibility of a change in the corresponding mode shapes, and it can also be observed in the case of the upper eigenvalues.

Based on the concepts presented, a numerical procedure has been developed with the purpose of determining the critical value $T^{(1,2)}$ of $T_{\overline{c}}$, which consists in the replacement of the values of λ_2 and $\overline{c}$ into Eq. (21). In this case the value $T^{(1,2)} = 995.9135$ has been obtained.

Table 4 depicts the first two exact values of the frequency coefficient λ and the corresponding mode shapes of an $SS-SS$ beam with $\overline{c} = 0.5$, for different values of the restraint stiffness $T_{\overline{c}}$. It is observed that the modes which correspond to λ_1 for

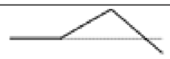
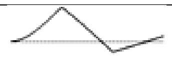
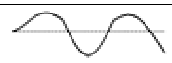
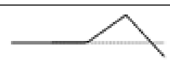
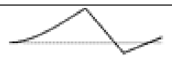
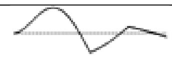
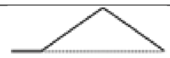
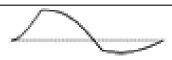
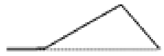
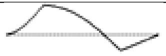
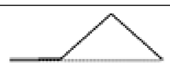
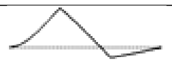
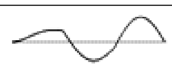
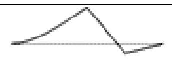
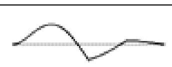
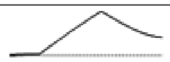
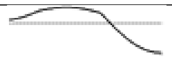
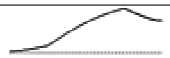
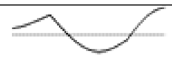
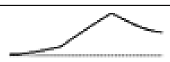
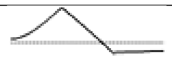
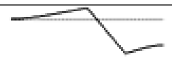
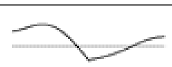
BC	$\bar{c}_1$	$\bar{c}_2$	λ_1		λ_2		λ_3	
$C-F$	0.2	0.6	1.438556		6.609714		9.735284	
	0.25	0.75	1.444605		5.385415		8.439022	
	1/3	2/3	1.505427		4.639534		10.705621	
	0.5	0.75	1.616772		3.382267		8.626570	
$SS-AA$	0.2	0.6	1.391308		6.521151		8.583106	
	0.25	0.75	1.413622		5.353031		8.319202	
	1/3	2/3	1.456066		4.624047		9.810502	
	0.5	0.75	1.564002		3.375018		8.611874	
$ER-ER$	0.2	0.6	3.200393		5.459548		6.224496	
	0.25	0.75	3.661568		5.060301		7.209206	
	1/3	2/3	3.508990		4.478442		6.445352	
	0.5	0.75	3.266975		4.056564		7.364875	

Table 2. Values λ_1, λ_2 and λ_3 of the frequency coefficient λ and mode shapes of a beam with two internal hinges with different non-symmetrical boundary conditions and values of the parameters $\bar{c}_i, i = 1, 2$. ($T_{\bar{c}_1} = T_{\bar{c}_2} = 1$). The case $ER - ER$ is defined by $T_L = R_L = 1000, T_R = R_R = 100$.

$T_{\bar{c}} = 0$ and $T_{\bar{c}} = T^{(1,2)} - \varepsilon$ are symmetric while the second mode are anti-symmetric. Nevertheless, when $T_{\bar{c}} = T^{(1,2)} + \varepsilon$, the first mode is anti-symmetric, and the second mode is symmetric. Obviously ε assumes a small value. It can also be observed that as $T_{\bar{c}}$ increases, the first modal shape presents inflection points as it is illustrated by the figure which corresponds to the case $T^{(1,2)} - \varepsilon$. The corresponding mode shapes are analog until $T_{\bar{c}} = T^{(1,2)}$. In this process we have that $\lambda_1 \to \lambda_2$ from the left $T_{\bar{c}}$ increases in the interval $\left[0, T^{(1,2)}\right]$. When $T_{\bar{c}} > T^{(1,2)}$ there is a change: the values of λ_1 remain constant meanwhile the values of λ_2 increase as $T_{\bar{c}}$ increases, and the original second mode $(T_{\bar{c}}) = 0$ becomes the new first mode, (i.e., the mode shape which corresponds to λ_1 when $T_{\bar{c}} > T^{(1,2)}$, is identical to the mode shape which corresponds to λ_2 when $T_{\bar{c}} = 0$.).

The described phenomenon can be generalized by arguing that there exists a critical value $T^{(i,i+1)}$ of $T_{\bar{c}}$ where $\lambda_i = \lambda_{i+1}, \forall i$. The equality of eigenvalues can be explained through the existence of roots of multiplicity of the frequency Eq. (21). The procedure to obtain the values $T^{(i,i+1)}$, is analogous to that used for $T^{(1,2)}$.

In order to analyse the variation of the parameters λ_2 and λ_3, it must be noted that the elastic restraint must be located at the point which coincides with the node of the modal shape which corresponds to λ_3. In consequence, it is necessary to adopt $\bar{c} = 1/3$. By applying the procedure described above the value $T^{(2,3)} = 3354.9547$ has been obtained. Table 5 depicts the corresponding values and mode shapes.

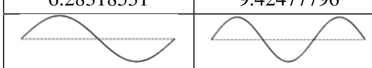
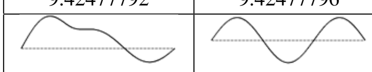
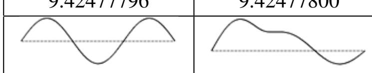
$T_{\bar{c}}$	λ_2	λ_3
0	6.28318531	9.42477796
		
$T^{(2,3)} - \varepsilon$	9.42477792	9.42477796
		
$T^{(2,3)} = 3354.9547$	9.42477796	9.42477796
$T^{(2,3)} + \varepsilon$	9.42477796	9.42477800
		

Table 5. Values λ_2, λ_3 of the frequency coefficient λ and mode shapes of a $SS - SS$ beam with $T^{(2,3)} = 3354.9547$ and $\bar{c} = 1/3$.

In the cases that correspond to a beam that is clamped

BC	$T_{\overline{c}_1}$	$T_{\overline{c}_2}$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
$SS-SS$	1	100	1.45611719	4.64282187	9.42477796	10.70691056	12.94235190	18.84955592
	100	100	4.31972616	4.90844437	9.42477796	10.74621939	12.98905869	18.84955592
	1000	100	4.55718873	7.66530463	9.42477796	11.07323213	13.48364215	18.84955592
	1000	1000	7.16349829	8.01393524	9.42477796	11.58970605	13.81448359	18.84955592
$C-SS$	1	100	4.28396838	4.93096581	9.81676729	11.83889375	13.76143072	19.22906792
	100	100	4.49969419	5.68516462	9.83627402	11.83894768	13.81211148	19.23144015
	1000	100	4.55795734	8.20313432	10.08227546	11.83943444	14.30200370	19.25413332
	1000	1000	7.28320226	8.44540826	10.19995164	12.40106641	14.46057634	19.26220167
$ER-ER$	1	100	4.25888244	4.74542202	6.69509498	8.93580767	11.66964943	15.09815518
	100	100	4.53063529	5.41229247	6.74864331	8.96276595	11.70947787	15.09913507
	1000	100	4.57054340	6.49115080	7.96847577	9.35593518	12.12807968	15.10746783
	1000	1000	5.57262634	7.47074935	8.42342470	9.99835788	12.28490310	15.24185358

Table 3. First six exact values of the frequency coefficient $\lambda$of a beam with two internal hinges, different boundary conditions, and intermediate points elastically restrained against translation located at $\overline{c}_1 = 1 / 3$ and $\overline{c}_2 = 2/ 3$. The case $ER - ER$ is defined by $T_L = R_L = 1000$, $T_R = R_R = 100$.

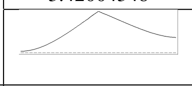
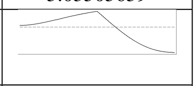
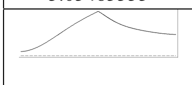
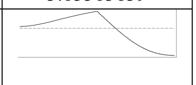
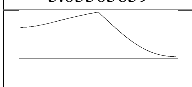
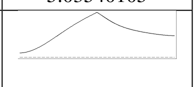
$T_{\overline{c}}$	λ_1	λ_2
0	3.42004348	5.03505659
		
$T^{(1,2)} - \varepsilon$	5.03483553	5.03505659
		
$T^{(1,2)} = 302.4782$	5.03505659	5.03505659
$T^{(1,2)} + \varepsilon$	5.03505659	5.03540105
		

Table 6. Values λ_1, λ_2 of the frequency coefficient λ and mode shapes of an $ER - ER$ beam ($T_L = R_L = 1000$, $T_R = R_R = 100$) with a free hinge located at $\overline{c}_1 = 0.5$. ($T^{(1,2)} = 302.4782$, $\overline{c}_2 = 0.617515$).

at $x = 0$ and simply supported at $x = l$ and to a beam with ends elastically restrained against rotation and translation ($T_L = R_L = 1000$, $T_R = R_R = 100$), the pairs λ_1, λ_2, λ_2, λ_3 and λ_3, λ_4 show the same features as those cases described previously. For brevity, the corresponding tables and figures are not included.

It is possible here to use the property of the boundary value problem described in Section 2 which allows to include one hinge located at $\overline{c}_1$ and an elastic restraint located at a different point $\overline{c}_2$. Table 6 depicts the exact values λ_1 and λ_2 of the frequency coefficient λ and the corresponding mode shapes of an $ER-ER$ beam ($R_L = T_L = 1000$, $R_R = T_R = 100$) with a free hinge located at $\overline{c}_1 = 0.5$ and the elastic restraint located at $\overline{c}_2 = 0.617515$ with $T^{(1,2)} = 302.4782$. The phenomenon of change of frequencies and mode shapes is analogous to the case that corresponds to a beam without hinge.

5. CONCLUSIONS

Hamilton's principle has been rigorously applied to obtain the boundary value problem and particularly the transition conditions, of a beam with two arbitrarily located internal hinges, four intermediate elastic support and ends elastically restrained against rotation and translation. Also a simple and accurate approach has been developed for the determination of natural frequencies and the mode shapes of free vibration. The mathematical model allows the inclusion of a hinge located at a point c_i and a translational restraint located at a different point c_j. This property permits to study the influence on frequencies and mode shapes of varying intermediate supports, located at the nodal points of higher modes. It has been demonstrated that the existence of a critical value of the dimensionless restraint parameter, which determines a particular behaviour of frequency parameters and the presence of mode shape switching. It has also been demonstrated that the eigenvalues and mode shapes show some of the same features when there are internal hinges.

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