This paper investigates the forced Van der Pol oscillator from a viewpoint of vibration flow and energy exchange. The method of averaging and the method of harmonic balance are used to derive analytical formulations of time-averaged power flow variables associated with steady-state periodic and quasi-periodic responses, respectively. Effects of bifurcations on time-averaged power flow are investigated both by analytical methods and numerical integrations. It is found that the time-averaged input power by the external harmonic force may become negative at some excitation frequencies. Correspondingly, the damping nonlinearity may lead to positive time-averaged dissipated power. The frequency band where there is negative time-averaged dissipated power is formulated through analytical approximations. It is also found that bifurcations of the forced oscillator from periodic to quasi-periodic responses may not lead to jump phenomenon in time-averaged input power. The work provides some new understanding of the power flow characteristics of the Van der Pol oscillator and demonstrates potential benefits of using such systems for vibration energy harvesting.

Keywords: Van der Pol oscillator, Power flow analysis, Nonlinear damping, Bifurcation, Quasi-periodic motion

1. Introduction

The Van der Pol oscillator is a typical dynamical system characterised by possible negative damping and has been shown to exhibit nonlinear complex behaviour. It was firstly proposed about a century ago to model the limit cycle oscillation (LCO) of currents in electrical circuits with a triode valve [1]. The model has then been frequently used to reveal the dynamics of various systems in many scientific and industrial fields. For example, it provided a good model for the gastric mill central pattern generator in lobsters [2]. The Van der Pol equation was also used to develop a model of the interaction between two plates in geological fault [3]. Similar oscillations that are exhibited by the VDP oscillator were also encountered in aero-elastic systems, such as in-flow bluff bodies [4].

The dynamic behaviour of the Van der Pol oscillator has been extensively investigated. However, most previous work were focused on the velocity and displacement response characteristics. The vibration power flow behaviour of the oscillator has not been fully clarified. Vibration power flow analysis (PFA) approach is a widely-accepted tool to characterise the dynamic behaviour of complex systems and coupled structures [5, 6, 7]. Vibration power flow combines the effects of force and
velocity in a single quantity, and thus can better reflect vibration transmission between sub-systems within an integrated system. Since the introduction of power flow concept, various PFA methods have been proposed to investigate linear vibration control systems.

It should be noted that although significant advances have been made on power flow analysis of linear dynamical systems, effective methods are needed to reveal the fundamental mechanisms governing vibration power input, dissipation and transmission in a nonlinear system. In particular, the power flow behaviour of nonlinear systems when exhibiting complex nonlinear phenomena, such as bifurcation and chaos, remains unclear. In view of this, there is a growing interest in developing PFA approach for nonlinear vibrating systems. It has been employed to reveal power flow characteristics of the Duffing oscillator [8], nonlinear isolation systems [9, 10], nonlinear vibration absorbers [11], vibration energy harvesting systems [12], coupled oscillators with nonlinear coupling interface [13] and some other nonlinear systems [14].

The current paper examines a typical oscillator, the Van der Pol oscillator, from the vibration power flow perspective so as to reveal the intrinsic power flow input and dissipation characteristics arising from the damping nonlinearity. The remaining content of the paper is organised as follows. The power flow formulations of the forced oscillator and numerical investigations of the effects of bifurcation on power flow are firstly presented Section 2. Section 3 will focus on the derivation of time-averaged power flow variables associated with periodic and quasi-periodic responses, and possible negative time-averaged input and dissipated powers. Section 4 provides some case study results and discussions. Conclusions are provided at the end of the paper.

2. Power flow formulations and numerical investigations

The general governing equation of a harmonically excited Van der Pol oscillator can be expressed as

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + \beta x = f \cos \omega t,$$  

(1)

where $f$ and $\omega$ denote the amplitude and frequency of the harmonic excitation force, respectively. Parameter $\alpha$ is assumed to be positive throughout this paper. The nonlinearity of this oscillator is demonstrated by the damping term, i.e., $f_d = \alpha(x^2 - 1)\dot{x}$. This expression and Fig. 1(a) show that the damping coefficient $\alpha(x^2 - 1)$ is positive when $|x| > 1$, i.e., the displacement is large. It becomes zero when $|x| = 1$ and negative when the absolute value of displacement $x$ is smaller than unity.

Multiplying by velocity $\dot{x}$ on both sides of Eq. (1), the equation of power balance is obtained, i.e.,

$$\ddot{x}\dot{x} + \alpha(x^2 - 1)\dot{x}^2 + \beta x\dot{x} = \dot{x}f \cos \omega t,$$  

(2)

which may alternatively be written as

$$\dot{K} + \dot{U} + p_d = p_{in},$$  

(3)

where $\dot{K} = \dot{x}\dot{x}$, $\dot{U} = \beta x\dot{x}$, $p_d = \alpha(x^2 - 1)\dot{x}\dot{x}$, $p_{in} = \dot{x}f \cos \omega t$ are the instantaneous rates of change of kinetic and potential energies and instantaneous dissipated and input powers, respectively. It can be seen that the dissipated power $p_d$ is positive when $|x| > 1$ and $x \neq 0$, as shown in the expression of the dissipated power and Fig. 1(b). When the displacement is large, the nonlinear damping term in the oscillator dissipates vibration energy, similar to the effects of conventional linear viscous damping. However, when $|x| < 1$ and $\dot{x} \neq 0$, the value of $p_d$ would be negative, indicating net energy input by the nonlinear force term. In other words, the nonlinear damping force plays the role of energy supply when the displacement is small. This feature of the Van der Pol oscillator differs greatly from linear vibration systems and has a significant impact on the system’s dynamic behaviour.

For a system with parameters set as $\alpha = 0.5$, $\beta = 1$, $f = 1$, Figs. 2(a)-(c) show a bifurcation diagram, the samplings of instantaneous input power $p_s$ with a sampling period of $2\pi/\omega$ and the variations of the time-averaged input power $\overline{p_{in}}$ averaged over an excitation period $2\pi/\omega$ in the steady
state motion, respectively. It’s shown that the system bifurcates from periodic to non-periodic motions when the excitation frequency is located at 0.54 or 1.28. In between these two critical frequencies, the system exhibits periodic motion. The peak value of the time-averaged input power is also encountered in this frequency range. It is also observed that the value of $\bar{p}_m$ may become negative at some excitation frequencies. Further investigations of the response frequency components by employing Fast Fourier Transform (FFT) show that the system exhibit quasi-periodic motions in the low and the high frequency ranges.

3. Analytical approximations of time-averaged power flows

3.1 Time-averaged associated with periodic responses

For periodic motions of the forced system, the displacement and velocity responses are assumed to be harmonic with the same frequency as that of the excitation, $x = r_1 \cos(\omega t + \phi), \dot{y} = -\omega r_1 \sin(\omega t + \phi)$. The method of averaging is now employed to find the time-averaged power flow variables. By rewriting Eq. (1) in the first order form and following the method of averaging, we have

$$
\dot{r}_1 \cos(\omega t + \phi) - r_1 \dot{\phi} \sin(\omega t + \phi) = 0, \quad -\dot{r}_1 \omega \sin(\omega t + \phi) - r_1 \dot{\phi} \omega \cos(\omega t + \phi) = f_1,
$$

where $f_1 = f \cos \omega t + \alpha \omega r_1 \sin(\omega t + \phi)(r_1^2 \cos^2(\omega t + \phi) - 1) + r_1 (\omega^2 - \beta) \cos(\omega t + \phi)$. The rates of changes of $r_1$ and $\phi$ can be obtained and are approximated by their average values over an excitation cycle:

$$
\dot{r}_1 = -\frac{\omega}{2\pi} \int_0^{2\pi} f_1 \sin(\omega t + \phi) \, dt, \quad \dot{\phi} = -\frac{\omega}{2\pi} \int_0^{2\pi} \frac{f_1}{r_1 \omega} \cos(\omega t + \phi) \, dt.
$$

Completing the integrations and setting $\dot{r}_1 = \dot{\phi} = 0$, we have

$$
\frac{f}{\sin \phi + \alpha \omega r_1 (\frac{r_1^2}{4} - 1)} = 0, \quad \frac{f}{\cos \phi + r_1 (\omega^2 - \beta)} = 0.
$$

A further simplification of these two equations to eliminate the sine and cosine terms yields

$$
f^2 = (\alpha \omega r_1)^2 (\frac{r_1^2}{4} - 1)^2 + r_1^2 (\omega^2 - \beta)^2.
$$

Figure 1: Variations of (a) nonlinear damping force and (b) dissipated power, with respect to displacement and velocity ($\alpha = 1$).
Figure 2: Effects of bifurcation on power flows. (a) Bifurcation diagram, (b) $p_s$ and (c) $p_{in}$.

A stability analysis can be performed and it can be shown that a solution with $r_1 \geq 2$ will always be stable. Using Eqs. (5), first-order approximations of the time-averaged input power (TAIP) and the time-averaged dissipated power (TADP) of the system over an excitation cycle ($T = 2\pi/\omega$) are formulated as

$$p_{in} = -\frac{1}{T} \int_0^T \omega r_1 f \cos \omega t \sin (\omega t + \phi) \, dt = -\frac{\omega r_1 f}{2} \sin \phi,$$

(7a)

$$p_d = \frac{1}{T} \int_0^T \alpha \omega^2 r_1^2 (r_1^2 \cos^2 (\omega t + \phi) - 1) \sin^2 (\omega t + \phi) \, dt = \frac{1}{8} \alpha \omega^2 r_1^2 (r_1^2 - 4),$$

(7b)

respectively. It shows that the expressions for time-averaged input and dissipated powers are identical. Also, both time-averaged input and dissipated powers become negative when $r_1^2 < 4$, i.e., the amplitude $r_1$ is smaller than 2.

### 3.2 Time-averaged associated with quasi-periodic responses

In the above approximation of the system’s power flows associated with periodic motions, only the response component with the same frequency as the excitation is considered. However, as shown in Fig. 2, the system can display quasi-periodic motion over a large frequency range. For analytical approximations of the corresponding power flows, the displacement and velocity responses are assumed to contain both the natural frequency $\omega_p$ for the free vibration component and the excitation frequency $\omega$ for the forced vibration component:

$$x(t) = \dot{x}_1 \cos \omega_p t + \dot{x}_2 \sin \omega_p t + \ddot{x}_1 \cos \omega t + \ddot{x}_2 \sin \omega t,$$

(8a)

$$\ddot{x}(t) = -\omega_p \dot{x}_1 \sin \omega_p t + \omega_p \dot{x}_2 \cos \omega_p t - \omega \ddot{x}_1 \sin \omega t + \omega \ddot{x}_2 \cos \omega t.$$

(8b)
The nonlinear term is expressed in a truncated Fourier form. Inserting the resultant expression as well as Eqs. (8) to the governing equation (1) and applying the harmonic balancing conditions yield the following equations

\[ (\beta - \omega_p^2)\ddot{z}_1 - \alpha_1 \omega_p \dot{z}_2 + \frac{1}{4} \alpha_1 \omega_p \dot{z}_2 (\ddot{z}_1^2 + \ddot{z}_2^2 + 2\dot{z}_1^2 + 2\dot{z}_2^2) = 0, \]  
\[ (\beta - \omega_p^2)\ddot{z}_2 + \alpha_1 \omega_p \dot{z}_1 - \frac{1}{4} \alpha_1 \omega_p \dot{z}_1 (\ddot{z}_1^2 + \ddot{z}_2^2 + 2\dot{z}_1^2 + 2\dot{z}_2^2) = 0, \]  
\[ (\beta - \omega^2)\ddot{x}_1 - \alpha \omega \dot{x}_2 + \frac{1}{4} \alpha \omega \dot{x}_2 (2\ddot{x}_1^2 + 2\ddot{x}_2^2 + \dot{x}_1^2 + \dot{x}_2^2) = f, \]  
\[ (\beta - \omega^2)\ddot{x}_2 + \alpha \omega \dot{x}_1 - \frac{1}{4} \alpha \omega \dot{x}_1 (2\ddot{x}_1^2 + 2\ddot{x}_2^2 + \dot{x}_1^2 + \dot{x}_2^2) = 0. \]  

Letting \[ \rho_1 = \sqrt{\ddot{z}_1^2 + \ddot{z}_2^2} \] and \[ r_1 = \sqrt{\ddot{x}_1^2 + \ddot{x}_2^2} \] and simplifying Eqs. (9a) and (9b) leads to

\[ (\beta - \omega_p^2)\rho_1^2 = 0, \quad \frac{1}{4} \alpha \omega_1 \rho_1^2 (4 - \rho_1^2 - 2r_1^2) = 0. \]  

Also, a manipulation of Eqs. (9c) and (9d) yields

\[ (\beta - \omega^2)r_1^2 = f \ddot{x}_1, \quad \frac{1}{4} \alpha \omega r_1^2 (2\rho_1^2 + r_1^2 - 4) = f \ddot{x}_2. \]  

When \[ r_1^2 > 2, \] Eqs. (10) requires that \[ \rho_1^2 = 0, \] i.e., the response contains no free vibration component and the motion will be periodic with the same frequency as the excitation. This case has been analysed earlier and the time-averaged power flow variables are formulated in Eqs. (7). When \[ r_1^2 < 2, \] previous analysis has shown that the periodic motion will be unstable and consequently there will be positive solutions of \( \omega_p \) and \( \rho_1 \) to Eqs. (10), i.e.,

\[ \omega_p = \sqrt{\beta}, \quad \rho_1^2 = 4 - 2r_1^2. \]  

In this situation, the system will exhibit quasi-periodic motion containing both the free vibration component with a fixed frequency of \( \omega_p = \sqrt{\beta} \) and the forced vibration component with frequency \( \omega \). Note that Eqs. (11) and (12) can be simplified further by eliminating \( \ddot{x}_1, \ddot{x}_2, \) and \( \rho_1^2 \) into the form:

\[ (\beta - \omega^2)^2 r_1^2 + \frac{\alpha^2 \omega^2 r_1^2}{16} (4 - 3r_1^2)^2 = f^2. \]  

The instantaneous input power corresponding to quasi-periodic responses can be approximated by

\[ p_{in} = f \cos \omega t (-\omega_p \dot{z}_1 \sin \omega_p t + \omega_p \dot{z}_2 \cos \omega_p t - \omega \dot{x}_1 \sin \omega t + \omega \dot{x}_2 \cos \omega t), \]  

which contains frequency components at \( \omega \pm \omega_p, \) 0 and 2\( \omega \). When the averaging time is set much larger than the periods of the frequency components in the response, only the stationary component of \( p_{in} \) will contribute to the time-averaged input power. Thus

\[ \bar{p}_{in} \approx \frac{\omega \ddot{x}_2 f}{2} = \frac{1}{8} \alpha \omega^2 r_1^2 (4 - 3r_1^2). \]  

where Eqs. (11) and (12) were used for simplifications.

### 3.3 On negative time-averaged input / dissipated power

Due to the damping nonlinearity in the oscillator, it has been shown previously that the instantaneous dissipated power of the system may become negative when the displacement is small. Similarly,
for time-averaged power flows, Figs. 2 and 3 as well as Eq. (7b) suggest that the time-averaged dissipated power $\bar{p}_d$ may become negative at some excitation frequencies. It would be of interest to investigate the frequency ranges where $\bar{p}_\text{in} = \bar{p}_d < 0$ using the analytical formulations. Note that from Eqs. (7) and (15), for negative values of $\bar{p}_\text{in}$ and $\bar{p}_d$, it requires $2 < r_1^2 < 4$ for stable periodic motions and $\frac{3}{4} < r_1^2 \leq 2$ for quasi-periodic motions. When the corresponding frequency-response relations described by Eqs. (6) and (13) are used, we have $\frac{\beta^2}{4} < (\omega^2 - \beta)^2 < \frac{3\beta^2}{4}$, which is equivalent to

$$\beta - \frac{\sqrt{3}f}{2} < \omega^2 < \beta - \frac{f}{2} \quad \text{or} \quad \beta + \frac{f}{2} < \omega^2 < \beta + \frac{\sqrt{3}f}{2}.$$  \hspace{1cm} (16)

It indicates that the location of frequency range for negative time-averaged input/dissipated power is only a function of the excitation amplitude $f$ and stiffness parameter $\beta$, independent of damping parameter $\alpha$. It can be shown that the value of $\bar{p}_\text{in}$ will be negative at the bifurcation point (with $r_1^2 = 2$) where the system response bifurcates from periodic to quasi-periodic motions. For both types of the motions, the power flow variables at the bifurcation frequency are

$$\bar{p}_\text{in} = \bar{p}_d = -\frac{1}{2}\alpha\omega^2.$$  \hspace{1cm} (17)

where Eqs. (7) and (15) are used. Clearly when a first-order analytical approximation is used, the variations of $\bar{p}_\text{in}$ and $\bar{p}_d$ at the bifurcation point will be smooth without jumps.

### 4. Results and discussions

For a system with $\alpha = 0.5, \beta = 1, f = 1.0$, Fig. 3 shows the corresponding time-averaged input power obtained by using analytical approximations and numerical simulations. For the latter, the averaging time is taken as 240 excitation cycles, which is found to be sufficiently large. It shows that the numerical results agree well with analytical approximations for periodic motion when the excitation frequency $\omega$ is close to the peak frequency. Away from the resonant frequency range, analytical formulations for quasi-periodic responses provide better estimations of power flows. It shows that the analytical approximation curves for periodic and quasi-periodic responses intersect at points $A''$ and $B''$. This is because at these points, $r_1$ equals $\sqrt{2}$, so that the value of $\bar{p}_\text{in}$ derived using either Eqs. (7) for periodic responses or Eq. (15) for quasi-periodic responses are the same.

![Figure 3: Power flow behaviour of the forced oscillator (a) verification and (b) effects of $\beta$. Solid and dashed lines: analytical approximation of periodic and quasi-periodic responses, respectively; circles: numerical results.](image)

The effects of parameter $\beta$ on the time-averaged input power (TAIP) are examined in Fig. 3(b). For clarity, the unstable branches of analytical approximation of periodic motions with $r_1^2 < 2$ are not shown in the figures. For a system with $\alpha = 0.5, f = 1.0$, Fig. 3(b) shows an increase in the stiffness
coefficient $\beta$ shifts the peak in each curve to the higher-frequency range. Also, the peak value of $p_{in}$ increases with $\beta$ while the minimum value decreases with an increasing $\beta$. A growing difference between analytical and numerical results is observed as $\beta$ increases. In the high-frequency range, the time-averaged input power becomes less sensitive to variations of parameter $\beta$.

5. Conclusions

This paper examined a typical oscillator with damping nonlinearity, i.e., the Van der Pol oscillator, from vibration power flow viewpoint. The effects of the damping nonlinearity on vibration power flows were investigated using both analytical approximations and numerical simulations. For the former, the method of harmonic balance and the method of averaging were employed while for the latter, the fourth-order Runge-Kutta method was used. The influences of different system parameters on time-averaged input power of the system were examined. Based on this study, it was found that the forced Van der Pol oscillator may either exhibit periodic motion when the excitation frequency $\omega$ is close to the natural frequency of the corresponding linear system, or quasi-periodic motion when $\omega$ is located in the high or low frequency ranges. The associated time-averaged input and dissipated powers are formulated analytically using the method of harmonic balance and the method of averaging. It was shown that there may be negative time-averaged input power and positive time-averaged dissipated power in a band of excitation frequencies. Analytical formulation showed that the band is only a function of the excitation amplitude $f$ and the stiffness parameter $\beta$, independent of the damping parameter $\alpha$. It was also analytically and numerically shown that bifurcations from periodic to quasi-periodic motions may not lead to jumps in time-averaged input / dissipated power curves.

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