VIBRATION REDUCTION BY USING NONLINEAR ACOUSTIC METAMATERIALS

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Linear acoustic metamaterials (LAMMs) are artificial periodic media structured on a size scale smaller than the wavelength of external excitation and have been widely used to manipulate sound in recent years. However, for vibration control, it is still a challenge to achieve good control performance at low frequencies using LAMMs. The work is intended to further lower and broaden the band gap of flexural wave propagation in beam by using the idea of nonlinear acoustic metamaterials (NAMMs). For this purpose, a one-dimensional NAMM chain is designed firstly, whose subwavelength meta-cell consist of a cubic-quintic Duffing oscillator and a linear oscillator coupled to a vibro-impact system, and thus a meta-cell of nonlinear dynamic characters can be obtained due to internal collision. Then, the displacement transmissibility and invariant manifolds of simplified mode are analysed, the results show that nonlinear mode can suppress resonance. Furthermore, the dispersion properties of beam are studied based on the homotopy analysis method which can filter the unstable multiple periodic solutions generated by harmonic balance method and is more accurate. The dispersion properties of NAMMs indicate that NAMMs have broader band gap compared with LAMMs, and the width of band gap can be further enhanced by increase nonlinearity, especially by increasing cubic nonlinear stiffness. However, the minimum frequency of band gap also increases, which can be avoided by quintic nonlinearity. For cubic-quintic NAMMs, lightweight, low-frequency and broadband characteristics are compatible and therefore can be achieved simultaneously for NAMs. These characteristics make NAMs an appropriate candidate for vibration reduction. Finally, the performance optimization of the NAMMs is discussed as well, where the influence of attached mass ratio on enhancing width of band gap is considered. The optimization results show the width of band gap can be improved by choosing appropriate attached mass ratio and increasing quintic nonlinear stiffness.

Keywords: nonlinear acoustic metamaterials, ultra-low, ultra-broad

1. Introduction

The nonlinear acoustic metamaterials (NAMMs) have the capability to reduce flexural wave propagation in an ultra-low and ultra-broad band that consists of bandgaps and chaotic bands [1]. They have several advantages over the linear acoustic metamaterials (LAMMs), for example, a broader band gap and a low frequency for LAMMs [2], so it is still a challenge to achieve good control performance at low frequencies using LAMMs. By contrast, the dispersion characteristic of NAMMs enables them to achieve low frequency and broadband band gaps simultaneously [2]. Regarding this, a one-dimensional NAMM chain whose subwavelength meta-cell consists of a cubic-quintic Duffing oscillator and a linear oscillator coupled to a vibro-impact system is designed and its characteristics are discussed in this paper.
2. MODEL AND BASIC DYNAMICS

In the one-dimensional basic model of a NAMM, the nonlinear oscillators with cubic-quintic stiffness \( f_{nl} = k_1 x + k_2 x^3 + k_3 x^5 \) are attached to the one-dimensional linear chain, as shown in Fig. (1).

![Figure 1: Basic model of a 1D NAMM.](image)

Defining \( u \) and \( y \) as displacement of the linear and nonlinear oscillators in each cell, respectively, with the Bloch theorem \( u_{n+1} = u_n \exp(-i\kappa a) \) of periodic structures, the system motion equations can be written by

\[
\begin{align*}
\omega^2 u''(\tau) + \alpha \omega u(\tau) &= \lambda \beta_1 (y - u) + \lambda \beta_2 (y - u)^3 + \lambda \beta_3 (y - u)^5 \\
\omega^2 y''(\tau) &= -\beta_1 (y - u) - \beta_2 (y - u)^3 - \beta_3 (y - u)^5
\end{align*}
\]

(1)

The parameters are defined as follows: \( \tau = \omega \tau, \omega_0 = \sqrt{k_0/m}, \lambda = m_0/m, \beta_1 = k_1/m_0, \beta_2 = k_2/m_0, \beta_3 = k_3/m_0, p = k \alpha, \) and \( \kappa \) is a wave vector, \( \alpha \) represents the lattice constant \( \alpha = \omega_0 \sqrt{2(1 - \cos p)}, \) and the generalized frequency is \( \Omega = \omega/\omega_0. \) The prime denotes the differentiation with respect to \( \tau. \) The excitation displacement \( u_{lb}(t) = A_0 \sin \omega \tau. \) In the simulations below, the five cases with different \( \beta_2 \) and \( \beta_3 \) are employed. The parameters are \( \lambda = 0.5, \beta_1 = 15\pi, A_0 = 0.005; L, \beta_2 = \beta_3 = 0; N1, \beta_2 = 10^4, \beta_3 = 0; N2, \beta_2 = 10^4, \beta_3 = 0; N3, \beta_2 = 10^4, \beta_3 = 10^5; N4, \beta_2 = 10^6, \beta_3 = 10^8. \) Here L denotes linear and N denotes nonlinear.

2.1 Nonlinear dynamics of simplified cells

A 2DOF simplified model is shown in Fig. 2(a). To be consistent with the responses of NAMMs, the excitation displacement applied to the linear oscillator of the simplified modes is \( u_{lb}(t). \)

![Figure 2: (a) Simplified model of a single cell. (b) Frequency response of the simplified model](image)
With the harmonic balance method in Eq. (1), the steady amplitudes of the simplified model can be attained.

\[
\begin{bmatrix}
K - \omega^2 M
\end{bmatrix} X + \frac{3}{4} B_{n1} (A - B)^3 + \frac{5}{4} B_{n2} (A - B)^5 = F_a
\]

\[
\beta_{n1} = \begin{bmatrix}
\lambda \beta_2 \\
-\beta_2
\end{bmatrix},
\beta_{n2} = \begin{bmatrix}
\lambda \beta_3 \\
-\beta_3
\end{bmatrix},
F_a = \begin{bmatrix}
\omega^2_0 A_0 \\
0
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
K = \begin{bmatrix}
0 & \lambda \beta_1 & -\lambda \beta_1 \\
-\beta_1 & \beta_1
\end{bmatrix}
\]

The solutions in the real domain express the fundamental nonlinear dynamic properties of the simplified model and the displacement transmissibility is defined as \( T_A = A_1/A_0 \). As illustrated in Fig. 2(b), the linear 2DOF system has two resonant modes corresponding to two linear modes (LMs). The nonlinearity caused by the cubic nonlinear stiffness \( \beta_2 \) and the quintic nonlinear stiffness \( \beta_3 \) has relatively less influence on the first LM but makes the second resonant LM disappear instead with nonlinear modes (NMs), which result in a broad range of weak response. Moreover, when \( \beta_2 \) is in the \( 10^4 \) order of magnitude, \( \beta_2 \) can further decrease the response of both the first and second resonant LM. However, when \( \beta_2 \) increases to the \( 10^6 \) order of magnitude, \( \beta_3 \) has less little influence on the transmissibility \( T_A \).

A nonlinear mode is defined as two-dimensional invariant manifold in phase space. The invariant manifolds of the simplified model in N3 is calculated. As illustrated in Fig. 3, the invariant manifold of a linear is a straight line. In contrast, the invariant manifolds of the two nonlinear modes are curves. Note that the chaotic motion of the second NM in a long time interval forms a region containing disorder curves. And for the system without damping, the lengths of the straight lines are infinite, but the volumes of the invariant manifolds of the two NMs are finite, which means that the motions are bounded. Therefore, the nonlinear modes have capacities to suppress the resonances.

![Figure 3: (a) First normal mode, (b) Second normal mode](image)

This capacity of cubic-quintic nonlinear stiffness to suppress resonance results from the nonlinear mode bifurcations and period-doubling bifurcations, which may cause chaotic responses in the systems. The mechanism is similar to the situation with only cubic nonlinear, researched by Fan [2].
3. Dispersion relation of the 1D metamaterial chain

3.1 Homotopy analysis method

Homotopy analysis method (HAM), first proposed by Liao [3], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [4,5]. Unlike previous analytic techniques, the HAM is independent of any small/large parameters and provides a convenient way to adjust and control the region and rate of convergence [6]. There exist some techniques to accelerate the convergence of a given series. Among them, the so-called homotopy Padé technique, which is a combination of the conventional Padé technique with the homotopy method, is widely applied [4,7]. Furthermore, HAM can filter the unstable multiple periodic solutions fixed by HBM and be more accurate [8]. This section details the analytical procedure of homotopy Padé technique for dispersion analysis.

For nonlinear metamaterial, to solve Eq. (2), the auxiliary linear and nonlinear operators are defined respectively as follows

\[ \mathcal{L}_u (f) = a_0^2 \left( \frac{d^2 f}{d \tau^2} + f \right) \]
\[ \mathcal{L}_y (f) = a_0^2 \frac{d^2 f}{d \tau^2}, \]
\[ \mathcal{N}_u [U, Y, \Lambda (q)] = \Lambda^2 (q) \frac{\partial^2 U (\tau, q)}{\partial \tau^2} + \alpha^2 U (\tau, q) \]
\[ - \lambda \beta_1 (Y - U) - \lambda \beta_2 (Y - U)^3 - \lambda \beta_3 (Y - U)^5 \]
\[ \mathcal{N}_y [U, Y, \Lambda (q)] = \Lambda^2 (q) \frac{\partial^2 Y (\tau, q)}{\partial \tau^2} + \beta_1 (Y - U) + \beta_2 (Y - U)^3 + \beta_3 (Y - U)^5 \]

The properties of the linear operators are
\[ \mathcal{L}_u (c_1 \sin \tau + c_2 \cos \tau) = 0 \]
\[ \mathcal{L}_y (c_3 \sin \tau + c_4) = 0 \] (4)

Then we can construct a homotopy
\[ \tilde{H}_u (U; q, h_1, H_1 (\tau)) = (1 - q) \mathcal{L}_u (U - u_0 (\tau)) - q h_1 H_1 \mathcal{N}_u [U, Y, \Lambda] \]
\[ \tilde{H}_y (Y; q, h_2, H_2 (\tau)) = (1 - q) \mathcal{L}_y (Y - y_0 (\tau)) - q h_2 H_2 \mathcal{N}_y [U, Y, \Lambda] \] (5)

Make \( \tilde{H}_u = \tilde{H}_y = 0 \), which leads to a zero-order deformation equation
\[ (1 - q) \mathcal{L}_u (U - u_0 (\tau)) = q h_1 H_1 \mathcal{N}_u [U, Y, \Lambda] \]
\[ (1 - q) \mathcal{L}_y (Y - y_0 (\tau)) = q h_2 H_2 \mathcal{N}_y [U, Y, \Lambda] \] (6)

Where the embedding parameter \( q \in (0, 1] \); \( u_0 (\tau) \) and \( y_0 (\tau) \) are initially guessed solutions of unknown parameters \( u (\tau) \) and \( y (\tau) \) respectively; \( h_1 \) and \( H_1 (\tau) \) are auxiliary parameters and functions that can adjust the convergence region and velocity of the homotopy series solutions; and \( \mathcal{L} (\cdot) \) and \( \mathcal{N} (\cdot) \) are linear and nonlinear operators, respectively, as defined in Eq. (4). The subscripts \( \| \) and \( \cdot \) represent the corresponding displacements.

In the case \( q = 0 \),
\[ U (\tau, 0) = u_0 (\tau), Y (\tau, 0) = y_0 (\tau), \Lambda_0 (\tau) = \omega_0 \] (7)
where \( \omega_0 \) denotes the zero-order approximation of \( \omega \). And when \( q = 1 \),
\[ U (\tau, 1) = u (\tau), Y (\tau, 1) = y (\tau), \Lambda (1) = \omega \] (8)
which means when \( q \) changes from 0 to 1, the approximate solutions \( U(\tau, q) \), \( Y(\tau, q) \) and \( \Lambda(q) \) change from initial guesses to their exact solutions \( u(\tau) \), \( y(\tau) \) and \( \omega \), respectively. All of them can be expanded with Taylor series in the zero domain of \( q \),

\[
U(\tau, q) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)q^m
\]

\[
Y(\tau, q) = y_0(\tau) + \sum_{m=1}^{+\infty} y_m(\tau)q^m
\]

\[
\Lambda(q) = \omega_0 + \sum_{m=1}^{+\infty} \omega_m q^m
\]

in which

\[
u_m(\tau) = \frac{1}{m!} \frac{\partial^m U(\tau, q)}{\partial q^m} \bigg|_{q=0}, \quad y_m(\tau) = \frac{1}{m!} \frac{\partial^m Y(\tau, q)}{\partial q^m} \bigg|_{q=0}, \quad \omega_m(\tau) = \frac{1}{m!} \frac{\partial^m \Lambda(q)}{\partial q^m} \bigg|_{q=0}
\]  

If the initial guesses, auxiliary parameters and auxiliary functions are chosen properly, series in Eq. (10) will converge at \( q = 1 \) and determine the exact solutions, which is proved by Liao [7].

The initial guesses are chosen as

\[
u_0(\tau) = A_0 \sin \tau, \quad y_0(\tau) = B_0 \sin \tau, \quad B_0 = \frac{\alpha^2 - \omega_0^2}{2\omega_0^2} A_0
\]  

where \( A_0 \) is also the parameter to control the elastic waves in the nonlinear metamaterials. First, differentiation \( m \) times Eq. (7) with respect to \( q \), then dividing by \( m! \), and finally setting \( q = 0 \), the \( m \)th-order deformation equation can be deduced

\[
\mathcal{L}_1[u_m(\tau) - \chi_m u_m(\tau)] = h_1 H_1(\tau) R_m^u[U, Y, \Lambda]
\]

\[
\mathcal{L}_2[y_m(\tau) - \chi_m y_m(\tau)] = h_2 H_2(\tau) R_m^u[U, Y, \Lambda]
\]

where

\[
\chi_m = \begin{cases} 0, & m = 1 \\ 1, & m > 1 
\end{cases}
\]  

and

\[
R_m^u[U, Y, \Lambda] = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}_u}{\partial q^{m-1}} \bigg|_{q=0}
\]  

\[
= \sum_{j=0}^{m-1} u_{m-1-j}(\tau) \left( \sum_{i=0}^{j} \omega_i \omega_{j-i} \right) + \alpha^2 u_{m-1}(\tau) - \lambda \beta_1 \left[ y_{m-1}(\tau) - u_{m-1}(\tau) \right] - \lambda \beta_2 \sum_{j=0}^{m-1} \sum_{i=0}^{j} \left[ y_{m-1-j}(\tau) - u_{m-1-j}(\tau) \right] y_i(\tau) - u_i(\tau) \left[ y_{j-i}(\tau) - u_{j-i}(\tau) \right] - \lambda \beta_3 \sum_{j=0}^{m-1} \sum_{i=0}^{j} \sum_{k=0}^{i} \left[ y_{m-1-j}(\tau) - u_{m-1-j}(\tau) \right] y_{j-i-k}(\tau) - u_{j-i-k}(\tau) \left[ y_{k-n}(\tau) - u_{k-n}(\tau) \right] - u_{k-n}(\tau) \left[ y_n(\tau) - u_n(\tau) \right]
\]  

\]
\[ R_m^[[U,Y,\Lambda]] = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}}{\partial q^{m-1}} \right|_{q=0} \]
\[ = \sum_{j=0}^{m-1} y_m^{n-j}(\tau) \left( \sum_{i=0}^{j} \omega_i \omega_{j-i} \right) \]
\[ + \beta_i \left[ y_m^{n-i}(\tau) - u_m^{n-i}(\tau) \right] \]
\[ + \beta_j \sum_{j=0}^{m-1} \sum_{i=0}^{j} \left[ y_m^{n-i}(\tau) - u_m^{n-i}(\tau) \right] y_i(\tau) \]
\[ - u_i(\tau) \left[ y_j(\tau) - u_j(\tau) \right] \]
\[ + \beta_j \sum_{j=0}^{m-1} \sum_{i=0}^{j} \sum_{k=0}^{i} \sum_{n=0}^{k} \left[ y_m^{n-i}(\tau) - u_m^{n-i}(\tau) \right] y_j(\tau) \]
\[ - u_j(\tau) \left[ y_j(\tau) - u_j(\tau) \right] \]
\[ - u_i(\tau) \left[ y_i(\tau) - u_i(\tau) \right] \]
\[ \text{(15)} \]

The subscript \( m \) denotes the \( m \)-th order variable. The solutions of Eq. (2) then can be approximated by the former \( N \)-th order at \( q = 1 \), which is called homotopy-series.

\[ u(\tau) = \sum_{m=0}^{N} u_m(\tau) \]
\[ y(\tau) = \sum_{m=0}^{N} y_m(\tau) \]
\[ \omega \approx \sum_{m=0}^{N} \omega_m \]
\[ \text{(16)} \]

To determine the coefficients \( c_i \) in Eq. (5), the initial boundaries of the high-order series are defined as

\[ u_m(0) = u_m'(0) = 0, \quad m \geq 1 \]
\[ \text{(17)} \]

For a nonlinear system containing cubic-quntic stiffness, the basis functions can be defined as the odd orders of sinusoidal functions

\[ \{ \sin \tau, \sin 3\tau, \ldots, \sin (2m-1)\tau \}, \quad m = 1, 2, 3 \ldots \]
\[ \text{(18)} \]

Therefore, \( R_m^u(\tau) \) and \( R_m^y(\tau) \) can be expressed as

\[ R_m^u(\tau) = a_{m,1} \sin \tau + \sum_{j=2}^{M} a_{m,j} \sin (2j-1)\tau \]
\[ \text{(19)} \]

Substituting Eq. (22) into Eq. (14), the relationship between the unknown \( (j-i)^{\text{th}} \)-order frequencies \( \omega_{j-i} \) between the parameters \( a_{m,j}, b_{m,j} \) can be found, which leads to the solution of \( \omega_{j-i} \). Furthermore, according to the properties of linear operator \( \mathcal{L}_u \) and \( \mathcal{L}_y \) described by Eq. (5), to avoid the secular terms \( \tau \cos \tau \) in \( u_m(\tau) \) and \( \tau^n \) in \( y_m(\tau) \), there must be \( a_{m,1} = 0 \) and \( c_3 = c_4 = 0 \) in Eq. (22) and Eq. (5), respectively. Moreover, the boundary condition \( u_m(0) = 0 \) leads to \( c_2 = 0 \) and the coefficient \( c_1 \) can be obtained with \( u_m'(0) = 0 \). In addition, since the expressions in Eq. (22) have already included all the basis functions defined in Eq. (16), and the solutions for the linear operators also
contain the basis functions, the auxiliary functions $H_1(\tau)$ and $H_2(\tau)$ can be defined as constants. For convenience and without losing universality, $H_1(\tau)$ and $H_2(\tau)$ are chosen as 1 in this paper.

However, if the convergence-control parameters $h_1$ and $h_2$ are chosen inappropriately, the Taylor series in Eq. (10) expanded in the zero domain converges slowly or even cannot converge at $q = 1$. The homotopy Padé approximant [7] provides the convergent solutions in a sufficiently large region. For convenience and without losing universality, $h_1$ and $h_2$ are chosen as 1 in this paper.

The $[m,n]$-th order Padé approximant is expressed as

$$\Lambda_{m,n}(q) = \left( \sum_{k=0}^{m} P_k q^k \right) / \left( 1 + \sum_{k=1}^{n} P_{m+1+k} q^k \right)$$

(20)

where $P_k$ depends on $\omega_m$. A rapidly convergent solution can be obtained by setting $q = 1$ in Eq. (23). Therefore, the $[m,n]$ Padé approximant of the frequency is $\omega^{(m,n)} = \Lambda_{m,n}(1)$. Furthermore, $\omega^{(2,2)}$ is sufficiently accurate for the nonlinear metamaterial mode, which can be described as

$$\omega^{(2,2)} = \omega_0 + \frac{\omega_0^2 (\omega_2 - \omega_1) + \omega_1^2 + \omega_0 \omega_1 (\omega_2 - 2\omega_1)}{\omega_2^2 + \omega_1^2 + \omega_1 (\omega_4 - \omega_3) - \omega_2 (\omega_4 + \omega_3)}$$

(21)

By iterative computation of Eq. (14), $\omega_m$ can be obtained and this process can be realized easily by Mathematica.

### 3.2 Dispersion relation

Dispersion relations of 1D metamaterials with different nonlinearity are calculated by homotopy analysis method in this paper.

![Dispersion curves of LAMMs and NAMMs](image)

Figure 4. (a) Dispersion curves of LAMMs and NAMMs, (b) width of bandgaps with different $\lambda$

For a cubic-quintic nonlinear NAMMs, its stop band is similar to that of the cubic stiffness nonlinear NAMMs where the dense resonances are significantly suppressed in the optical branch, which is defined as the OB band by Wen [2]. As shown in Fig. 4, when $\beta_2$ is in the $10^4$ order of magnitude, it can make the acoustic branch curve down. And on this basis, $\beta_3$ can further move down the acoustic branch and also move up the optical branch curve, resulting a broader total band gap (with OB band). However, further increasing $\beta_1$, for example, to the $10^6$ order of magnitude, the acoustic branch curve moves up, but the optical branch curve changes from a concave curve to a convex curve, which makes the band gap (with OB band) of NAMMs broader. However, on this basis, $\beta_1$ can only move the optical branch curve a little. The results indicate that the cubic nonlinear stiffness plays a
dominant role in generating a broad chaotic band gap, which identifies with the displacement transmissibility analysis in previous section. Note that although increasing $\beta_3$ can generate a broader band gap, the minimum normalized frequency of band gap also increases significantly, which is disadvantage to low-frequency application. Therefore, $\beta_3$ is helpful to generate a broad and low band gap.

As a structure parameter, the influence of $\lambda$ on width of band gap is studied in this paper. As illustrated in Fig. 4(b), for both LAMMs and NAMMs, the width of band gaps shift upward with increasing $\lambda$. However, the enhance performance of $\beta_3$ on band gaps vary in a nonmonotonic way. When $\lambda = 0.4$, $\beta_3$ has the best performance to enhance band gaps width by 8%.

4. Conclusion

A one-dimensional NAMM chain is designed in this paper, whose subwavelength meta-cell consist of a cubic-quintic Duffing oscillator and a linear oscillator coupled to a vibro-impact system. The displacement and invariant manifolds of simplified mode are analysed, the results show that the NM of NAMMs has capability to suppress resonance. Furthermore, process of homotopy analysis method for calculating dispersion properties of NAMMs is deduced. The dispersion properties show that increasing cubic nonlinear stiffness can make a broader bandgap. However, to decrease the minimum normalized frequency of band gap for low-frequency applications, the quintic nonlinear stiffness is necessary, which can lower the minimum frequency of band gap by 3.5% meanwhile broader the band gap by 8%. Finally, the influence of attached mass ratio is discussed and the result indicate that the width of band gap can be improved by choosing appropriate attached mass ratio and increasing quintic nonlinear stiffness.

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