The paper presents the results of numerical calculation of the vibrodiagnostic parameters of the presence of a semielliptical breathing crack in the structural bar elements of circular cross section. The vibration properties under forced vibrations and at the frequency of principal, super- and subharmonic resonances are considered as the vibrodiagnostic parameters of the presence of a crack. The authors provide the results of numerical calculations aimed at the determination of the influence of the place of vibration recording on the second harmonic under forced and resonance vibrations, as well as the relation between the amplitudes of the dominant harmonics of displacements and strains under the bar vibrations at the frequency of the principal resonance. Also, the effect of the force application along the bar length on the vibrodiagnostic parameters at super- and subharmonic resonances is shown. It is demonstrated that the crack can be located using the obtained dependencies.

Keywords: bar of circular cross section; semielliptical breathing crack; vibrodiagnostic parameter; super- and subharmonic resonances

1. Introduction

With advances in modern mechanical engineering the requirements for the machine components, which are both under development and in operation, are ever-growing. In the former case, it involves the problem of their lifetime enhancement task. In the later case, the focus is on the functional performance in a wide range of operational modes. However, experience has proven that in both cases it is impossible to avoid the occurrence of various damages such as fatigue cracks that are regarded as the most critical ones.

A crack in the structure affects its elastic properties and, consequently, the vibration properties, which variation is widely employed in the damage diagnostics [1 - 3], in particular determination of its dimensions and location. At present, all the attempts to find the ideal parameter to be used in the vibration diagnostics have not met with success. At this stage of development the most common parameters of the presence of a crack are the frequency and mode of vibrations [4, 5]. But these parameters are efficient only for open cracks, whereas for breathing ones their use is possible only for large size damages [6]. Thus, here the parameters recorded at the excitation of super-and subharmonic resonance modes [7, 8] are found to serve well as the vibrodiagnostic parameters.

The goal of the paper lies in the investigation of the influence patterns of a semielliptical breathing fatigue crack on the properties of the forced flexural vibrations of the bar of circular cross section, as well as in the search for the efficient vibrodiagnostic parameters of its presence.
2. **Object of investigation and its modeling**

The paper considers the bar of circular cross section of $L=230 \text{ mm}$ in length and of $R=10 \text{ mm}$ in radius having a semielliptical breathing crack with the dimensions of $a=10 \text{ mm}$, $b=4 \text{ mm}$ at the distance of $x_c=0.1L$ from restraining (Fig.1). The harmonic force $P = P_0 \sin(\nu t)$ was applied at the distance of $x_p$ from restraining, where $P_0$ and $\nu$ are the force amplitude and frequency, respectively. The mechanical properties of the specimen materials are the following: $E = 200 \text{ GPa}$, $\rho = 7800 \text{ kg/m}^3$, while the logarithmic decrement of vibrations is $\delta = 0.01$.

![Figure 1: A cantilever bar of circular cross section with a semielliptical breathing crack.](image)

3. **Problem solution algorithm**

The problem of the determination of the properties of forced vibrations of the bar with a breathing crack has been solved numerically consisting of four main stages.

1. Development of the finite element model for the object under investigation complying with the procedures [9 - 10].

   The bar is divided into equal finite elements with the region in the crack vicinity characterized by mesh refinement. The breathing crack is represented in the form of a mathematical cut with mutual non-penetration of its faces ensured by the contact task to be solved using three-dimensional elements.

2. Determination of the displacement of arbitrary bar points within the specified time interval.

   The nonlinear differential equation is used to describe the forced vibrations of the bar with a crack under the action of harmonic force $P(t)$:

   $$\begin{bmatrix} M \end{bmatrix} \dddot{\{u\}} + \begin{bmatrix} D \end{bmatrix} \ddot{\{u\}} + \begin{bmatrix} K \end{bmatrix} \{u\} = \{P(t)\},$$  

   (1)

   where $\begin{bmatrix} M \end{bmatrix}$ and $\begin{bmatrix} D \end{bmatrix}$ are the inertial and dissipative matrices of the system, respectively; $\begin{bmatrix} K \end{bmatrix}$ is the stiffness matrix varying in time depending on the contact interaction between the crack faces and defining the system nonlinearity under consideration; $\{u\}$, $\{\dot{u}\}$, $\{\ddot{u}\}$ are the column vectors of displacement, velocity, and acceleration, respectively; $\{P(t)\}$ is the column vector of the external harmonic loading.

   Equation (1) is solved using the Newmark method.
where $\gamma$ and $\beta$ are the Newmark integration parameters determining accuracy and stability of integration.

The point is that the overall time $T$ of forced vibrations of the bar is divided into $N$ steps at $\Delta t = T/N$ starting from their excitation to the attainment of the specified mode under initial conditions $\{\dot{u}\}_0 = 0$, $\{\ddot{u}\}_0 = 0$, $\{\dddot{u}\}_0 = 0$. Then, considering the solution for the preliminary time values at each $N$ step, the approximate values for $u$ are defined for each time moment $0, \Delta t, 2\Delta t, \ldots, T$, which allows one to obtain the time-dependent displacement $u(t)$.

4. The Fourier analysis.

A significant feature of the nonlinear bar system with a breathing crack is the possibility of occurrence of nonlinear resonances and higher harmonics in the displacement at harmonic excitation. To obtain the displacement amplitudes $A_k$, where $k$ is the harmonic number, the fast Fourier transformations (FFT) of the obtained displacement dependencies for the chosen point of the cantilever bar $u(t)$ are used.

5. Data processing and analysis of results.

4. Results of investigations

The numerical calculations were performed to determine the influence of the semielliptical breathing fatigue crack location on the vibrodiagnostic parameters of its presence in the cantilever bar of circular cross section.

Let us consider the forced subresonance ($\omega = 240\, \text{Hz}$) and resonance vibrations ($\omega = 262.5\, \text{Hz}$) of the bar of circular cross section with a semielliptical breathing crack. At the amplitude of the force $P_0=10\, \text{N}$ the bar displacement was recorded at nine nodes along the bar. Figure 2 illustrates the dependencies of the second harmonic of forced $A_{2,\alpha}$, and resonance $A_{2,\beta}$ vibrations on the point of data recording using the performed numerical calculation results. As is seen from the data, within the crack section one observes a break point on the curve of the dependencies obtained at the forced vibrations, while at the resonance vibrations there is a jump on the curve.

![Figure 2: Dependencies of the second harmonic at forced (a) and resonance (b) vibrations of the bar with a semielliptical breathing crack in the section $x_T = 0.1L$ on the point of vibration recording.](image)
Let us consider the possibility of using the ratio between the constant component \( A_0 \) and the first harmonic \( A_1 : \frac{A_0}{A_1} \), as well as the ratio between the amplitudes of dominant harmonics (second and first) \( \frac{A_2}{A_1} \), as the vibrodiagnostic parameter of the presence of a breathing crack at the principal resonance in the calculation of displacements and similar ratios in the recording of strain amplitudes \( \frac{(A_{01})_{st}}{(A_{2})_{st}} \). Figures 3 and 4 depict the calculation results. Noteworthy is that the displacement amplitude was determined for the nodes on the neutral bar line, whereas the strain amplitude for the nodes on the surface with a crack was determined on the bar line opposite to it.

![Graph](image1)

**Figure 3:** Dependencies of the ratio between the constant component (■) and the second displacement harmonic amplitude (●) to the amplitude of the principal harmonic on the point of vibration recording.

![Graph](image2)

**Figure 4:** Dependencies of the ratio between the constant component (α) and the second strain harmonic amplitude (β) to the amplitude of the principal harmonic on the point of recording on the surface with a crack (■,●) and opposite to it (□,○).

The analysis of the dependencies in Fig. 3 implies that the use of \( \frac{A_{01}}{} \) in the diagnostics of the presence of a crack is quite possible due to the break curve point and the minimum values of the specified characteristic in the section with damage. The relation between the amplitudes of the do-
dominant harmonics $\bar{A}_2$ remains to be conjectural. This is explained by the fact that a drastic increase of the specified relation between the amplitudes (its maximum value is not outlined) is responsible for the crack location.

A more accurate recording of some damage is possible in the determination of the longitudinal bar strain. It is evident that in the section with a crack there are the maximum values of $(\bar{A}_{01})_s$ and $(\bar{A}_2)_s$. Moreover, on the surface with a crack their values are significantly higher as compared with those on the opposite surface.

Aside from the occurrence of nonlinear harmonics in the spectrum of forced and resonance vibrations, the system with a breathing crack is characterized by the so-called nonlinear resonance modes, at which the excitation frequency is a multiple of the natural frequency of vibrations of the object under investigation. Thus, excitation of vibrations at the frequency that is two-times lower as compared with the resonance frequency $\nu = 0.5f_0$ causes the excitation of the superharmonic resonance, while at the frequency, which is two-times higher, it leads to the subharmonic resonance $\nu = 2f_0$. Even if the amplitude of vibrations at given resonance modes is considerably lower as compared with that at the principal resonance, the spectrum of its vibrations involves the vibration harmonic, which is consistent with their excitation frequency, and the harmonics that are consistent with the principal resonance frequency. The latter can be several times higher than the harmonic value at the excitation frequency. The specified harmonics can be detected in the increase of the amplitude of the excitation force. The ratios between the amplitudes of the dominant harmonics, in particular the second (resonating) amplitude and the first under superharmonic resonance, as well as the first (resonating) amplitude and the second under subharmonic resonances, might be used as the vibrodiagnostic parameters of the crack presence.

The calculations of the vibrodiagnostic parameters of the crack presence under super- and subharmonic resonances were performed for the bar of circular cross section under conditions of variation of the point of application of the harmonic force $x_p$ along the bar length. Here the place of vibration recording remained unchanged: $x = L$.

Figure 5 provides the results of calculations in the form of dependencies of the vibrodiagnostic parameters $\bar{A}_{2/1}$, $\bar{A}_{1/2}$ of the presence of a crack under super- and subharmonic resonance on the place of force application.

![Figure 5](image)

Figure 5: Dependencies of the amplitudes of the dominant displacement harmonics under super-(■) and subharmonic (●) resonance on the point of force application.

From the dependencies in the figure it can be concluded that the nonlinearity of the vibration process, due to the presence of a breathing fatigue crack, depends greatly on the point of force ap-
plication, which can be employed to locate the crack. Therefore, under superharmonic resonance the crack location falls within the minimum value of the vibrodiagnostic parameter, while under subharmonic resonance it complies with the maximum value of the vibrodiagnostic parameter.

5. Conclusions

Based on the numerical calculations aimed at the determination of the mechanisms of influence of the semielliptical breathing fatigue crack location on the properties of the forced flexural vibrations of the bar of circular cross section it was established that the most efficient vibrodiagnostic parameter of the presence of breathing cracks is the ratio between the amplitudes of the dominant displacement harmonics under super- and subharmonic resonances, as well as the ratio between the constant component or the amplitude of the second strain harmonic and the amplitude of the first harmonic under the principal resonance. One of the specific features of the mentioned vibrodiagnostic parameters is their ability to imply the crack location.

REFERENCES