NONLINEAR WAVES IN DISPERSIVE VISCOUS MEDIA

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Solutions to nonlinear hyperbolic systems describing weakly nonlinear quasitransverse waves in weakly anisotropic elastic media are studied. The influence of small-scale effects of dissipation and dispersion are analyzed. Small-scale processes determine a discontinuity structure and a set of discontinuities with stationary structures. Among discontinuities with stationary structures there are special ones on which (in addition to relations following from conservation laws) some additional relations should be satisfied which follow from the requirement for the discontinuity structure to exist. On the phase plane the structure of such discontinuity is represented by an integral curve connecting to saddle points.

Keywords: shocks with a stationary structure, special shocks

1. Introduction

The problem of an adequate mathematical description of nonlinear waves in continuous media with account of the influence of real factors plays a central role in mechanics. Practice demands lead to increasing the complexity of mathematical models, which must take into account the complex interaction of diverse processes in solid media. Classical models for studying waves are complicated by taking into consideration new physical effects. Studying the behavior of such complex media, which is described of high-order equations, requires new methods and tools of analysis.

An important stage in the study, which is necessary in many cases, is leaving the framework of hyperbolic equations and using more complex equations in order to investigate the structure of discontinuities and select a unique solution in the cases where the solution of the corresponding hyperbolic system is nonunique.

This paper is devoted to studying problems related to the propagation of the one-dimensional nonlinear waves in elastic media under various additional assumptions about processes going on in the shock wave structure and other high-gradient layers. We describe the complex behavior of nonlinear waves in other cases investigated previously. The discovered special features of solutions are not related to any specifics of the equations of elasticity theory; they are generic for enough hyperbolic systems of equations. In many cases, small-amplitude nonlinear waves are described by universal hyperbolic equations, such as the Hopf equation.

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In many cases, small-amplitude nonlinear waves are described by universal hyperbolic equations, such as the Hopf equation. Small-scale processes are usually described by the Burgers, KdV, or KdV-Burgers equations. The terms responsible for dispersion processes can be written in various manners. Equations supplemented with terms taking into account small-scale processes describe the structure of discontinuities (shocks) and determine a set of discontinuities with a stationary structure. As was noted in [1, 2, 3], the presence of dispersion terms with second derivatives implies that the governing parameters of the medium include a pseudovector. If the latter is lacking, the dispersion terms of the lowest order of differentiation are third-order terms [4, 5]. For these two models of dispersion effects, a set of discontinuities with a stationary structure was found in [4, 5]. Moreover, it was shown that this set includes special discontinuities, whose number increases indefinitely as the dispersion parameter grows (in the viscous case). The nonlinearity parameter has a large effect on the behavior of simple and shock waves and, as a consequence, on solutions of standard self-similar problems.

2. Nonlinear waves with close characteristic velocities

To approximately describe waves corresponding to characteristics with close velocities, we use the hyperbolic system of equations derived in [6]. This system describes nonlinear quasi-transverse waves in a weakly anisotropic elastic medium. Waves in which the displacements of the particles are approximately parallel to surfaces of constant phase are called quasi-transverse. The system of equations has the form

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial R(u_1, u_2)}{\partial u_\alpha} \right) = 0, \quad \alpha = 1, 2$$

(1)

$$R(u_1, u_2) = \frac{1}{2}f(u_1^2 + u_2^2) + \frac{1}{2}g(u_2^2 - u_1^2) - \frac{1}{4}\kappa(u_1^2 + u_2^2)^2, \quad f, g, \kappa = \text{const}$$

(2)

Here, $u_\alpha$ are the components of the shear strain of the medium, $u_\alpha = \partial w_\alpha/\partial x$, $w_\alpha$ is the displacement of a medium particle in the direction of the Cartesian axes $x_\alpha$ ($\alpha = 1, 2$) parallel to constant-phase planes, and $x$ is the Lagrangian coordinate of a medium particle. The solution is a function of $x$ and time $t$ ($u_\alpha = u_\alpha(x, t)$). The constant $g$ is the anisotropy parameter, $\kappa$ is a constant characterizing nonlinear effects, and $f$ is the characteristic velocity (doubled) in the absence of nonlinearity and anisotropy (i.e., for $\kappa = 0$, $g = 0$). The sign of $\kappa$ has a large effect on the behavior of quasi-transverse simple and shock waves [7, 8].

System (1) expresses conservation laws and is associated with the jump conditions (see [8])

$$\left[ \frac{\partial R}{\partial u_\alpha} \right] - W[u_\alpha] = 0.$$  

(3)

where $W = dx/dt$ is the velocity of the discontinuity in the Lagrangian coordinate. The square brackets in (1) denote the jumps in the corresponding quantities across the discontinuity: $[u] = u^l - u^r$ (here, the indices $r$ and $l$ denote the values ahead of and behind the discontinuity, respectively). Eliminating $W$ from system (1) yields the Hugoniot equation (see [8])

$$\left( (u_1^l)^2 + (u_2^l)^2 - (u_1^r)^2 - (u_2^r)^2 \right) (u_1^l u_2^l - u_2^r u_1^r) + \frac{2g}{\kappa} (u_1^l - u_1^r)(u_2^l - u_2^r) = 0$$

(4)
By the Hugoniot curve, we mean the set of pairs of values $u^1_1, u^1_2$ for fixed $u^1_1, u^1_2$ and also the curve with coordinates $u^1_1, u^1_2$ in the plane $u_1, u_2$ passing through the initial point $A(u^1_1, u^1_2)$.

Discontinuous solutions must satisfy evolutionary conditions \[8\] ensuring that the problem of a discontinuity interacting with small perturbations is well-posed. Evolutionary conditions obtained when only the conservation laws hold at a discontinuity without imposing additional relations of different nature are called \textit{a priori evolutionary}.

The \textit{a priori evolutionary conditions} for jump relations \[3\] consist of two groups of inequalities, namely,

\[ c^r_2 \leq W, \quad c^l_1 \leq W \leq c^l_2, \]  
\[ c^r_1 \leq W \leq c^r_2, \quad W \leq c^l_1. \]  

and determine discontinuities known as fast and slow shock waves, respectively \[8\]. Here, $c^r_\alpha$ and $c^l_\alpha$ ($\alpha = 1, 2$) are the characteristic velocities of system \[1\] ahead of and behind the discontinuity, respectively. Expressions for the characteristic velocities were obtained in \[7\]:

\[ c_{1,2} = f - 2\kappa(u^2_1 + u^2_2) \mp \kappa \sqrt{(u^2_1 - u^2_2 - g/k)^2 + 4u^2_1u^2_2} \]  

Other evolutionary discontinuities can exist, which are called \textit{special} (see \[2\], \[3\], \[4\], \[5\], \[9\], \[10\], \[11\], \[12\]). To be evolutionary, they have to satisfy additional conditions for a stationary structure to exist. Moreover, the requirement for the existence of a structure may lead to some \textit{a priori} evolutionary discontinuities becoming nonevolutionary.

### 3. Problem of stationary shock structure and special discontinuities

To describe small-scale processes that are important inside a shock structure, Eqs. \[1\] have to be supplemented with terms responsible for dissipation and dispersion. Following \[4\], \[5\], we study equations of the form

\[ \frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial R(u_1, u_2)}{\partial u_\alpha} \right) - \mu \frac{\partial^2 u_\alpha}{\partial x^2} + m \frac{\partial^3 u_\alpha}{\partial x^3} = 0, \quad \alpha = 1, 2 \]  

\[ \mu, m = \text{const}, \quad \mu \geq 0, \quad m/\mu >> 1. \]

where $\mu$ is the viscosity and $m$ is the dispersion coefficient.

It was shown in \[4\] that the equations describing quasi-transverse waves in a nonlinear weakly anisotropic dielectric have the same form.

Consider the problem of a stationary shock structure. Solutions of system \[8\] are sought in the form $u_\alpha = u_\alpha(\xi)$, $\xi = -x + Wt$, such that, as $\xi \to -\infty$, the quantities $u_\alpha$ tend to $u^r_\alpha$ (corresponding to the state ahead of the shock) and, as $\xi \to +\infty$, they tend to $u^l_\alpha$ (corresponding to the state behind the shock, i.e., to a state represented by a point on the Hugoniot curve). Integrating system \[8\] once yields the following system of two second-order ordinary differential equations for the functions $u_\alpha(\xi)$:

\[ mu''_\alpha + \mu u'_\alpha = -\frac{\partial Z}{\partial u_\alpha}, \quad \alpha = 1, 2 \]  

\[ u'_\alpha = \frac{\partial u_\alpha}{\partial \xi}, \quad u''_\alpha = \frac{\partial^2 u_\alpha}{\partial \xi^2} \]  

\[ Z(u_1, u_2) = R(u_1, u_2) - \frac{1}{2} W(u^2_1 + u^2_2) + Q_1 u_1 + Q_2 u_2 \]  

\[ Q_\alpha = \text{const}, \quad \alpha = 1, 2 \]
The states at $\xi = -\infty$ and $\xi = +\infty$ are associated with singular points of system (9) that are determined by the equalities
\[ \frac{\partial Z}{\partial u_\alpha} = 0, \quad u'_\alpha = 0, \quad \alpha = 1, 2 \]  
(11)
The solution of the shock structure problem is represented to an integral curve joining one singular point to another. Note that
\[ \frac{d}{d\xi} \left( \frac{m}{2} (u'_1^2 + u'_2^2) + Z(u_1, u_2) \right) = -\mu \left( \left( \frac{du_1}{d\xi} \right)^2 + \left( \frac{du_2}{d\xi} \right)^2 \right) < 0. \]  
(12)
Equations (9) have the form of those describing a point particle of mass moving with friction in a potential force field with the potential energy $Z$. The variable $\xi$ plays the role of time. Inequality (12) means that the energy of the point particle is reduced due to friction. It implies that
\[ Z(u_1^l, u_2^l) < Z(u_1^r, u_2^r). \]  
(13)
Inspection of the left-hand sides of Eqs.(9) shows that, as $\mu \to 0$ and $m \to 0$ with $m/\mu^2 = \text{const}$ the shape of the structure is preserved (up to scale), while its width decreases, so that, in the limit, the solutions of system (9) become discontinuities.

System (9) has the fourth order, and the coordinates of its singular points are determined by system (11).

The function $Z(u_1, u_2)$ can have one, three, or five stationary points.

The stationary point of $Z(u_1, u_2)$ can be a minimum, saddle point, or maximum. Relying on the analogy with the motion of a point particle, we can easily determine the type of a singular point of system (9) in the four-dimensional phase space.

Let us examine the behavior of the integral curves in the case $c_1^r < W < c_2^r$. Figure 2 shows a version of level lines of $Z(u_1, u_2)$ and singular points for $u'_1 = 2, u'_2 = 0.4, f = 1, g = 1$ and $\kappa = 4$. Here, the initial point $A$ is a saddle, $S$ is a stable focus (a minimum of $Z(u_1, u_2)$), and the points $B$ and $D$ are unstable focuses (maxima of $Z(u_1, u_2)$). The discontinuity $A \to S$ is a priori evolutionary (see

![Figure 1: Level lines $Z(u_1, u_2) = \text{const.}$](image-url)
The discontinuities $A \rightarrow B$ and $A \rightarrow D$ are not possible, since inequality (13) does not hold.

As $\xi \to +\infty$, the integral curve leaving the point $A$ must enter one of the singular points $S, C$ or go to infinity. The arrival of the integral curve at the saddle point $C$ should be treated as an exceptional case, which is possible at the special velocity $W = W^*$. The value $W = W^*$ corresponds to the point $C$ on the Hugoniot curve. This point lies on an a priori nonevolutionary part of the Hugoniot, but, since it corresponds to the chosen value $W^*$, which should be regarded as an additional relation at the discontinuity, we conclude that the discontinuity $A \rightarrow C$ is evolutionary. Such discontinuities were called special (see [2, 3, 4, 5, 13]).

4. Conclusions

Stationary structures of discontinuities in nonlinear elastic weakly anisotropic media are studied. It is shown (in the framework of the model chosen) that the number of special discontinuities depends not only on small-scale processes of dissipation and dispersion, but on a sign of non-linearity parameter as well. An analysis of solutions to the problem showed that the behavior of the solutions are determined by small-scale processes.

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