PARAMETRIC VIBRATION ACTUATED BY FRINGING ELECTROSTATIC FIELDS

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In the Micro-Electro-Mechanical System (MEMS), the micro-resonator always needs a wider actuating frequency band, in order to improve its dynamic performance. Parametric vibration actuated by fringing electrostatic force exhibits rich nonlinear behaviours and has wide operation frequency bands. Since when the thickness of the stationary electrode is far greater than that of the movable cantilevered beam or the slit gap between the two is relative small, the relationship between the electrostatic force and the displacement is nonlinear. This paper proposes a new fitting formulation on the electrostatic force and the displacement near the equilibrium point of the cantilevered beam. This parameter model includes linear and cubic stiffness. Then the governing equation and the dynamic characters of movable cantilevered beam which actuated by the nonlinear electrostatic force are investigated. The results show that as the initial displacement, the length of the beam and the actuating voltage rise, the effects of the nonlinear parameters of the fringing electrostatic force on the dynamic behaviours will enhance. When the actuating frequency of the fringing electrostatic force is close to the double of the resonance frequency of the cantilevered beam, the fringing electrostatic force can actuate the beam. Moreover, this will give nonlinear system more resonance opportunities than that of linear system. Such results are useful to expand the actuating frequency range of the resonator.

Keywords: parametric vibration, fringing electrostatic, nonlinear

1. Introduction

A wider frequency band will significantly improve the performance [1] of the gyroscope [2, 3], energy harvesting [4, 5], filter [6, 7] and logic device [8, 9]. Some new driving modes were proposed to broaden the bandwidth of the resonator. Among them, many parametric vibrations were introduced [10-13], since they may have wide operation frequency bands.

The electrostatic actuation method is an important method in Micro Electro-Mechanical Systems (MEMS) [14], and the system actuated by the electrostatic force is parametric excitation system [12, 15, 16]. Among the electrostatic actuation structures, parallel-plates configuration is the most used in MEMS [17-19]. However, this actuation method leads to instabilities like pull-in. In order to obtain large displacement amplitude and avoid the instabilities, the fringing-field actuation technique gets attention. Slava Krylov et al. (2011) modelled the curved micro beam which were actuated by electrodes located out of the plane, and analysed the snap-through behaviour of this micro beam [20]. Slava Krylov et al. (2013) then performed experimental and theoretical analysis of micro-cantilevers actuated by fringing-field electrostatics. This actuation approach has several advantages, including
the ability to obtain large amplitude displacements without the limitation by the proximity of the electrodes, and the possibility of significantly tuning the resonant frequency response range [21]. Quoc Chi Nguyen et al. (2015) proposed a closed-loop control for a parametrically excited micro-cantilever beam, to provide the desired vibrational amplitude even in the case of wide variation of the system parameters [22]. Hassen Ouakad et al. (2016) studied the different behaviour of curved micro beam with low initial elevation and relative high initial elevation, when actuated with fringing-field electrostatic force [23]. Shahrzad Towfighian et al. (2017) demonstrated the effect of a flexible support on a parametrically excited electrostatic MEMS resonator. This resonator can use weaker electrostatic fringe fields to get higher vibrational amplitude [10]. Naftaly Krakover et al. (2017) designed a bi-stable cantilever, which was actuated by a fringing-field electrostatic force and a linearly increasing deflection-independent “mechanical” uniformly distributed transversal force [24]. These researches show that the statics and dynamics actuated by the fringing electrostatic force can reveal many new behaviours and phenomena.

In the present work, some researchers always care about the situation that the distance between the movable beam and the stationary electrode is relative large [24]. And when the thickness of the movable beam is near that of the stationary electrode, the relationship between the electrostatic force and the displacement is considered to be nearly linear [21]. However, when the thickness of the stationary electrode is far greater than that of the movable beam, or the slit gap is relative small, the relationship between the electrostatic force and the displacement is nonlinear. Therefore, the nonlinear behaviours should be analysed in this case.

This study provides an analytical investigation of the parametric vibration actuated by fringing electrostatic fields. Compared with linear relationship between the electrostatic force and the displacement, the nonlinear relationship has more accurate analysis results. Furthermore, the nonlinear dynamic character makes the resonator widen frequency band.

2. Problem formulation

In Figure 1, $l_b$, $w_b$ and $t_b$ denote the length, width and thickness of the movable cantilevered beam respectively; $l_s$, $w_s$ and $t_s$ denote the length, width and thickness of the stationary electrode, respectively; and $d_g$ is the slit gap between the movable beam and stationary electrode; $d$ is the initial displacement of the movable beam measured from the stationary electrode.

![Figure 1 Schematic illustration of micro-cantilevered beam](image)

Make $u(x,t)$ denote the displacement of the beam. According to the elastic beam theory, the governing equation and the boundary conditions of the beam are

$$EI \frac{\partial^4 u(x,t)}{\partial x^4} + \rho w_s l_b \frac{\partial^2 u(x,t)}{\partial t^2} = q_e + q_a$$

(1)

$$u(0,t) = \frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial^2 u(l_s,t)}{\partial x^2} = \frac{\partial^3 u(l_s,t)}{\partial x^3} = 0$$

(2)

where $E$ is the Young’s modulus; $l = w_b t_b^2/12$ is the inertial moment of the beam cross section; $\rho$ is the mass density of the beam; $q_e$ represents the excitation force per unit length; $q_a$ represents the aerodynamic force per unit length.
The distributed electrostatic force $q_e$ which is generated on the basis of the fringing fields, is given by [22]
\[ q_e = f_e \cdot \nu_e^2 \]  
(3)
where $f_e$ is electrostatic force on the movable beam per unit length per square voltage; $\nu_e(t)$ is a combined DC/AC voltage actuating the micro-beam, i.e., $\nu_e(t) = \nu_{DC} + \nu_{AC}\cos\omega_c t$.

Figure 2 shows the relationship between the electrostatic force and the displacement. Among them, (a) shows this relationship is nearly linear, when the initial displacement $d$ is less than the slit gap $d_g$, (b) shows this relationship is nonlinear, when the thickness of the stationary electrode is far greater than that of the movable cantilevered beam, or the slit gap $d_g$ is relative small.

![Figure 2](image)

Figure 2 Electrostatic force on the movable beam

Then the fitting formulation on the electrostatic force are
\[ f_e = e_p (1 + e_p u) \] (a)
\[ f_e = e_p (1 + e_p u + e_p u^2 + e_p u^3) \] (b)

where $e_p$, $e_{p1}$, $e_{p2}$ and $e_{p3}$ are fitting parameters.

The distributed aerodynamic force $q_a$ is [25,26]
\[ |q_a| = 0.5 \rho_a W_b C_a \left( \frac{\partial u(x,t)}{\partial t} \right)^2 \]  
(5)
where the direction of the aerodynamic force is opposite to the direction of the velocity; $c_a$ is the drag coefficient; $\rho_a$ is the density of the air.

For analytical convenience, we introduce the non-dimensional variables.
\[ X = \frac{x}{l_b}, T = \omega t, \omega = \frac{1}{l_b^2} \sqrt{\frac{EI}{\rho_w l_b}}, U = \frac{u}{d}, C_{op1} = e_p d, C_{op2} = e_p d^2, C_{op3} = e_p d^3, \]
\[ C_a = \frac{\rho c_a d}{2 \rho t_b}, W_e = \frac{\omega}{\omega_v}, V_{AC} = \frac{v_{AC}}{v_{DC}}, Q_{ep} = \frac{l_b^4 e_p v_{DC}^2}{E l_d}. \]  
(6)

When the relationship between the electrostatic force and the displacement is nonlinear, by substituting the non-dimensional variables into the governing equation, the dimensionless governing equation and the corresponding boundary conditions are
\[ \frac{\partial^4 U}{\partial X^4} + \frac{\partial^2 U}{\partial T^2} = (1 + C_{op1} U + C_{op2} U^2 + C_{op3} U^3) Q_{ep} \left( 1 + 2 V_{AC} \cos\omega_c T \right) - C_a \frac{\partial U}{\partial T} \left| \frac{\partial U}{\partial T} \right| \]  
(7)
\[ U(0,T) = \frac{\partial U(0,T)}{\partial X} = 0, \quad \frac{\partial^2 U(1,T)}{\partial X^2} = \frac{\partial^2 U(1,T)}{\partial X^3} = 0. \]  
(8)
The steady-state solution of the dimensionless governing equation is written by $U(X, T) = \Phi(X)\Theta(T)$, and the mode function $\Phi(X)$ for cantilevered beam is given by

$$\Phi(X) = \text{ch}\lambda X - \cos\lambda X + \xi (\text{sh}\lambda X - \sin\lambda X)$$

(9)

where $\lambda = 1.875$, $\xi = - (\text{ch}\lambda + \cos\lambda)/(\text{sh}\lambda + \sin\lambda)$.

Then the governing equation is

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Theta}{\partial T^2} \Phi = \left(1 + C_{op}\Phi + C_{op2}\Phi^2 + C_{op3}\Phi^3\right)Q_{op} (1 + 2V_{ac}\cos W_{T}) - C_{e} \left(\frac{\partial\Theta}{\partial T}\right)\frac{\partial\Theta}{\partial T}$$

(10)

Multiplying the outcome by the mode function, and integrating the resultant equation from $X = 0$ to 1, then an ordinary differential equation with respect to time is obtained as

$$\frac{\partial^2\Theta}{\partial T^2} + \alpha_x\Theta = \left(1 + \alpha_{op}\Theta + \alpha_{op2}\Theta^2 + \alpha_{op3}\Theta^3\right)\alpha_{fp} (1 + 2V_{ac}\cos W_{T}) - \alpha_e \left(\frac{\partial\Theta}{\partial T}\right)\frac{\partial\Theta}{\partial T}$$

(11)

in which the coefficients are

$$\alpha_s = \int_0^1 \frac{\partial^2\Phi}{\partial X^2} \Phi dX \quad \alpha_{fp} = \int_0^1 \Phi dX \quad \alpha_{op} = \int_0^1 C_{op}\Phi dX \quad \alpha_{op2} = \int_0^1 C_{op2}\Phi^2 dX \quad \alpha_{op3} = \int_0^1 C_{op3}\Phi^3 dX$$

$$\alpha_s = \int_0^1 \Phi dX$$

The displacement splits into a static displacement and a dynamic displacement, i.e. $\Theta(T) = \Theta_{op} + \Theta_{p}(T)$, and we obtain static equation and dynamic equation.

$$\alpha_e \Theta_{op} = \left(1 + \alpha_{op}\Theta_{op} + \alpha_{op2}\Theta_{op}^2 + \alpha_{op3}\Theta_{op}^3\right)\alpha_{fp}$$

(12)

$$\theta_{p} + \alpha_s\theta_{p} = \left(1 + \alpha_{op}\Theta_{op} + \alpha_{op2}\Theta_{op}^2 + \alpha_{op3}\Theta_{op}^3\right)2\alpha_{fp} V_{ac}\cos W_{T} +$$

$$\left[\alpha_{op}\theta_{p} + \alpha_{op2}\left(2\Theta_{op}\theta_{p} + \theta_{p}\right) + \alpha_{op3}\left(2\Theta_{op}\theta_{p} + 3\Omega_{op}\theta_{p} + \theta_{p}\right)\right]^{\alpha_{fp} \left(1 + 2V_{ac}\cos W_{T}\right) - \alpha_e \theta_{p}}$$

(13)

3. Vibration analysis

The dynamic equation is simplified, then we get

$$\theta_{p} + W_{p}\theta_{p} + eK_{p}\theta_{p} + eK_{p}\theta_{p} + e\alpha_{p}\theta_{p}\theta_{p}\theta_{p} = eF_{c} \cos W_{T} + e\left[K_{ep1}\theta_{p} + K_{ep2}\theta_{p} + K_{ep3}\theta_{p}\right]\cos W_{T}$$

(14)

in which the coefficients are

$$e = 1, \ (nW_{c})^2 = W_{c}^2 + 2\alpha_{e}, W_{c}^2 = \alpha_{p} - \alpha_{p3}\left(\theta_{p} + 2\alpha_{p2}\Theta_{op} + 3\alpha_{p3}\Theta_{op}\right)$$

$$K_{p} = -\alpha_{p} \left(\alpha_{p2} + 3\alpha_{p3}\Theta_{op}\right), \ K_{p} = -\alpha_{p} \left(\alpha_{p2} \Theta_{op}\right), \ F_{c} = 2\alpha_{fp} V_{ac} \left(1 + \alpha_{op}\Theta_{op} + \alpha_{op2}\Theta_{op}^2 + \alpha_{op3}\Theta_{op}^3\right)$$

$$K_{ep1} = 2\alpha_{fp} V_{ac} \left(\alpha_{p1} + 2\alpha_{p2}\Theta_{op} + 3\alpha_{p3}\Theta_{op}\right), \ K_{ep2} = 2\alpha_{fp} V_{ac} \left(\alpha_{p2} + 3\alpha_{p3}\Theta_{op}\right), \ K_{ep3} = 2\alpha_{fp} V_{ac} \alpha_{p3}.$$
where $\alpha_0$ is the amplitude of $\vartheta_p$, $\beta_0$ is the phase difference with $W_e$, $(\;)'$ denotes the derivate with respect to $T_1 = \varepsilon T$.

The steady-state periodic motion corresponds to the solution of the system of equations, by conditions $\alpha_0' = 0$ and $\beta_0' = 0$, we obtain

\[
\left( \frac{2\alpha W^2 \alpha_0'}{\pi F_e} \right)' + \left( \frac{K_\alpha \alpha_0' - 2\sigma \alpha_0}{2F_e + K_{\alpha_0} \alpha_0'} \right)' = 0 \tag{17}
\]

\[
\tan \beta_0 = \frac{2\alpha W^2 \alpha_0'}{\pi F_e} + \frac{2F_e + K_{\alpha_0} \alpha_0'}{K_\alpha \alpha_0' - 2\sigma \alpha_0} \tag{18}
\]

The geometric parameters of the structure are $l_b = l_s = 50\text{mm}$, $w_b = 4\text{mm}$, $t_b = 0.1\text{mm}$, $w_s = 10\text{mm}$, $t_s = 1\text{mm}$, $d_g = 0.1\text{mm}$.

The fitting parameters of the nonlinear electrostatic force show in Table 1. Among them, the units of parameters are subject to the SI unit.

<table>
<thead>
<tr>
<th>$d$</th>
<th>0.1×$10^{-3}$</th>
<th>0.2×$10^{-3}$</th>
<th>0.3×$10^{-3}$</th>
<th>0.4×$10^{-3}$</th>
<th>0.5×$10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_p$</td>
<td>-0.0152×$10^{-7}$</td>
<td>-0.0511×$10^{-7}$</td>
<td>-0.1282×$10^{-7}$</td>
<td>-0.2673×$10^{-7}$</td>
<td>-0.4890×$10^{-7}$</td>
</tr>
<tr>
<td>$e_{pl}$</td>
<td>1.4527×$10^{-4}$</td>
<td>1.0392×$10^{-4}$</td>
<td>0.8165×$10^{-4}$</td>
<td>0.6620×$10^{-4}$</td>
<td>0.5519×$10^{-4}$</td>
</tr>
<tr>
<td>$e_{p2}$</td>
<td>6.7908×$10^{-7}$</td>
<td>4.0443×$10^{-7}$</td>
<td>2.4159×$10^{-7}$</td>
<td>1.5450×$10^{-7}$</td>
<td>1.0557×$10^{-7}$</td>
</tr>
<tr>
<td>$e_{p3}$</td>
<td>2.2636×$10^{-11}$</td>
<td>0.6741×$10^{-11}$</td>
<td>0.2684×$10^{-11}$</td>
<td>0.1287×$10^{-11}$</td>
<td>0.0704×$10^{-11}$</td>
</tr>
</tbody>
</table>

Based on Table 1, Figure 3 shows the effects of the initial displacement, the length of the beam and DC/AC voltage on the frequency response curve of the primary resonance. When these four parameters are relative small, the system exhibit linear-like behaviour. When these four parameters are big enough, the electrostatic force can lead to nonlinear vibration. Furthermore, as the increase of these four parameters, especially the increase of the length of the beam and DC voltage, the nonlinear vibration strengthens. Since a larger amplitude is always expected, these four parameters are always set big enough, and then the nonlinear vibration is unavoidable. At the same time, the nonlinear expands the actuating frequency range.

![Figure 3](image-url)

**Figure 3** Frequency response curve of the primary resonance under different initial displacement (a); under different length of the beam (b); under different DC voltage (c); under different AC voltage (d).

Setting $n=0.5$, the parametric resonance is analysed, we obtain
\[ -W \alpha_0' = \left( \frac{1}{2} K_{EP} \alpha_0 + \frac{1}{4} K_{EP} \alpha_0^3 \right) \sin 2 \beta_0 - \frac{1}{2\pi} \alpha_0 W^2 \alpha_0' \]  
\[ -W \alpha_0 \beta_0' = \left( \frac{1}{2} K_{EP} \alpha_0 + \frac{1}{2} K_{EP} \alpha_0^3 \right) \cos 2 \beta_0 - \frac{3}{4} K_\sigma \alpha_0^3 + \sigma \alpha_0 \]  

where \( \alpha_0 \) is the amplitude of \( \theta_p \), \( \beta_0 \) is the phase difference with \( W_0 \), \( (\cdot)' \) denotes the derivate with respect to \( T_1 = \epsilon T \).

The steady-state periodic motion corresponds to the solution of the system of equations, by conditions \( \alpha_0' = 0 \) and \( \beta_0' = 0 \), we obtain

\[ \left( \frac{2\alpha W^2 \alpha_0^2}{2\pi K_{EP} \alpha_0 + \pi K_{EP} \alpha_0^3} \right)^2 + \left( \frac{3K_\sigma \alpha_0^3 - 4\sigma \alpha_0}{2K_{EP} \alpha_0 + 2K_{EP} \alpha_0^3} \right)^2 = 1 \]  
\[ \tan 2\beta_0 = \frac{2\alpha W^2 \alpha_0^2}{2\pi K_{EP} \alpha_0 + \pi K_{EP} \alpha_0^3} \frac{2K_{EP} \alpha_0 + 2K_{EP} \alpha_0^3}{3K_\sigma \alpha_0^3 - 4\sigma \alpha_0} \]

When the actuating frequency is near the double of the resonance frequency, the beam also vibrates. Based on Table 1, Figure 4 shows the effects of the initial displacement, the length of the beam and DC/AC voltage on the frequency response curve of the parametric resonance. The same as the primary resonance, with the increase of these four parameters, especially the increase of the length of the beam and DC voltage, the nonlinear vibration strengthens. When these four parameters are relative small, the actuating frequency range is small. However, the beam has wide operation frequency bands, when these four parameters are relative big, since the nonlinear can expand the actuating frequency range.

Figure 4 Frequency response curve of parametric resonance under different initial displacement (a); under different length of the beam (b); under different DC voltage (c); under different AC voltage (d).

Based on Table 1, Figure 5 compares the primary resonance and the parametric resonance. We can find that the resonance frequency rises clearly, especially when \( V_{DC}=500V \). In addition, the nonlinear behaviour of the parametric resonance is stronger than the one of the primary resonance. Since the nonlinear behaviour can expand the actuating frequency range, it is useful to enlarge the working frequency of the resonator.
4. Conclusion

By electrostatic simulation, we can know that the fringing electrostatic force has nonlinear fitting parameter. Since this parameter model includes linear and cubic stiffness, the electrostatic force can lead to nonlinear vibration of the system. This paper presents the dynamic investigations of the micro-cantilevered beam actuated by fringing electrostatic force, and analyses the effects of the initial displacement, the length of the beam and the actuating voltage on the dynamic behaviours. The increase of the initial displacement, the length of the beam and actuating voltage, especially the increase of the length of the beam and DC voltage, make the nonlinear vibration strengthen. When the actuating frequency is near the double of the resonance frequency, the beam can also vibrate. Moreover, nonlinear system has more resonance opportunities than linear system. Such dynamic results are useful to enlarge the working frequency of the resonator.

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