ORTHOGONAL INTERACTION OF ALFVEN WAVES WITH FAST MAGNETOACOUSTIC WAVE IN HEAT-RELEASING FINITE CONDUCTIVE MEDIUM

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The orthogonal parametrical interaction between Alfven and fast magnetoacoustic waves in the heat-releasing conductive medium is considered. It has been shown that there is an exponential growth of Alfven wave amplitudes in the fully conductive medium without heat-release. Finiteness of conductivity leads to appearance of the parametric amplification threshold and bounds the parametric amplification time. Heat-releasing can lead to additional damping of fast magnetoacoustic waves and decreasing of parametric amplification time. However, in the case of isentropic instability, the heat-release leads to magnetoacoustic wave amplification. This amplification is accompanying by the bi-exponential growth of Alfven wave amplitudes due to the parametric energy transfer from unstable fast magnetoacoustic waves.

Keywords: MHD-waves, parametric interaction

1. Introduction

Energy transfer by Alfven waves from the lower solar atmosphere layers is considered as possible mechanism of coronal heating. The reason is that Alfven waves due their nature can propagate without sufficient damping into the solar corona where they can decay into magnetoacoustic (MA) waves and Alfven waves with lower frequencies. According to [1], the following fast damping of MA waves can lead to the heating of the solar corona. Moreover, the Alfven waves with the sufficient energy for the acceleration of the solar wind and coronal heating to the observed velocity and temperature have been already observed [2]. However, the mechanism of appearance of Alfvén waves with the observed energy remains unknown.

One of the possible mechanisms of such Alfven waves appearance is the parametric energy transfer from powerful magnetoacoustic waves to weak Alfven waves. Similar interaction was considered in [3, 4] where it was shown that amplification of Alfven waves due to parametric decay of magnetoacoustic waves propagating along magnetic field is possible in the isentropically unstable medium with dominated gasdymanic pressure. Another example of this parametric interaction is the so-called swing wave-wave interaction between fast magnetoacoustic waves orthogonal to the magnetic field and Alfven waves propagating along the field [5, 6]. Fast magnetoacoustic waves affects on the magnetic field component along which Alfven waves propagate. As a consequence fast magnetoacoustic wave affects on the phase speed of Alfven waves.

In this work, influence of heat-release and finite conductivity on the initial stage of swing wave-wave interaction is considered.
2. Mathematical formulation

Let us consider the fully ionized plasma medium described by the system of MHD equations (1):

\[
\frac{\partial \vec{B}}{\partial t} = \text{rot}[\vec{v} \times \vec{B}] + \frac{c^2}{4 \pi \sigma} \Delta \vec{B}, \quad \text{div}\vec{B}.
\]

\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla P - \frac{1}{4\pi} \left[ \vec{B} \times \text{rot} \vec{B} \right], \quad \frac{\partial \rho}{\partial t} + \text{div}\rho\vec{v} = 0,
\]

\[
C_{\text{ve}} \rho \left( \frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla) T \right) - \frac{k_B T}{m} \left( \frac{\partial P}{\partial t} + (\vec{v} \cdot \nabla) P \right) = -\rho \mathcal{X}(p,T) + \frac{j^2}{\sigma},
\]

\[
\mathcal{X}(p,T) = L(p,T) - Q(p,T), \quad P = \frac{k_B T \rho}{m}, \quad j = \frac{c}{4\pi} \nabla \times \vec{B}.
\]

In the Eqs. (1) \( \rho \), \( T \), and \( P \) are the density, temperature, and pressure in plasma, respectively; \( \vec{v} \) and \( \vec{B} \) are the vectors of the velocity and magnetic field; \( j \) is the current density vector; \( c \) is the speed of light; \( \sigma \) is the electric conductivity coefficient; \( k_B \) is the Boltzmann constant; \( C_{\text{ve}} \) is the high-frequency specific heat at constant volume; \( d \) \( \text{d}t = \partial / \partial t + \vec{v} \cdot \nabla \). Here, \( L(p,T) \) and \( Q(p,T) \) are cooling and heating rates, respectively; \( \mathcal{X}(p,T) \) is the generalized heat-loss function that is widely applied in studies of the thermal instabilities, beginning from groundbreaking works [7, 8]. It equals zero under steady-state conditions (i.e., \( \mathcal{X}(p_0, T_0) = 0 \)).

It is well known that magnetoacoustic and acoustic waves can be unstable and grow under the isentropic instability criterion [9–15]:

\[
\frac{\mathcal{X}_{0_0}}{(\gamma – 1)} + \mathcal{Z}_{0_0} < 0, \quad \mathcal{Z}_{0T} = \frac{T_0}{Q_0} \left( \frac{\partial \mathcal{X}}{\partial T} \right)_{\rho = p_0, t = T_0} > 0, \quad \mathcal{X}_{0_0} = \frac{\rho_0}{Q_0} \left( \frac{\partial \mathcal{X}}{\partial \rho} \right)_{\rho = p_0, t = T_0}.
\]

3. Orthogonal parametric interaction

3.1 Fast magnetoacoustic wave

Let us consider that the unperturbed magnetic field vector \( \vec{B}_0 \) is along the z-axis and the medium is bounded in the x-axis direction (\( I \) is the size of the medium along x-axis). The dependencies of medium parameters on y-coordinate are neglected (\( \partial / \partial y = 0 \)). In this case, it is possible to describe Alfven and magnetoacoustic waves separately in the linear approximation. For the fast magnetoacoustic wave orthogonal to the vector \( \vec{B}_0 \), the linearization procedure (\( \rho = \rho_0 + \rho', B_z = B_0 + b_z \), ..., \( \rho'/\rho_0 - b_z/B_0 = ... \approx 0 \)) of Eqs. (1) yields:

\[
\frac{\partial b_z}{\partial t} = -B_0 \frac{\partial v_z'}{\partial x} + \frac{c^2}{4\pi \sigma} \frac{\partial^2 b_z}{\partial x^2}, \quad \rho_0 \frac{\partial v_z'}{\partial t} = -\frac{\partial}{\partial x} \left( \frac{P' + B_0 b_z}{4\pi} \right) \frac{\partial \rho'}{\partial t} = \frac{\rho_0}{4\pi} \frac{\partial v_z'}{\partial x},
\]

\[
C_{\text{ve}} \rho_0 \frac{\partial T'}{\partial t} - \frac{k_B T_0}{m} \frac{\partial \rho'}{\partial t} + \rho_0 \mathcal{X}_{0_0} \rho' + \rho_0 \mathcal{Z}_{0T} T' = 0, \quad \frac{m}{k_B T_0} \frac{P'}{P} = \frac{m P_0}{k_B T_0} \rho'.
\]

Equations (3) can be reduced to one Eq. (4) which can be rewritten in more simple way in low-frequency (\( \omega \tau_0 << 1 \)) limit (see Eq. (5)) and high-frequency (\( \omega \tau_0 >> 1 \)) limit (see Eq. (6)).

\[
C_{\text{ve}} \left[ \frac{\partial^2 b_z}{\partial t^2} - \frac{\partial^2 b_z}{\partial x^2} \right] - \frac{c^2}{4\pi \sigma} \frac{\partial^3 b_z}{\partial x^2 \partial t} - \frac{c^2}{4\pi \sigma} \frac{\partial b_z}{\partial x} \frac{\partial^2 b_z}{\partial x^2} = \frac{c^2}{4\pi \sigma} \frac{\partial^3 b_z}{\partial x^2 \partial t} - \frac{c^2}{4\pi \sigma} \frac{\partial^2 b_z}{\partial x^2},
\]

\[
+ \frac{C_{\text{ve}}}{\tau_0} \left[ \frac{\partial}{\partial t} \left( \frac{\partial^2 b_z}{\partial t^2} - \frac{\partial^2 b_z}{\partial x^2} \right) - \frac{c^2}{4\pi \sigma} \frac{\partial^3 b_z}{\partial x^2 \partial t} - \frac{c^2}{4\pi \sigma} \frac{\partial^2 b_z}{\partial x^2} \right] = 0.
\]

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\[
\frac{\partial^2 b_y}{\partial t^2} - \left(c_0^2 + c_a^2\right) \frac{\partial^2 b_y}{\partial x^2} - \left(\frac{c_a^2 v_m}{c_0^2 + c_a^2} + \frac{\xi_0}{\rho_0}\right) \frac{\partial^3 b_y}{\partial x^2 \partial t} = 0,
\]  

(5)

\[
\frac{\partial^2 b_y}{\partial t^2} - (c_x^2 + c_a^2) \frac{\partial^2 b_y}{\partial x^2} - \frac{c_a^2 v_m}{c_x^2 + c_a^2} \frac{\partial^3 b_y}{\partial x^2 \partial t} + \frac{\xi_0 c_v^2}{\rho_0 (c_a^2 + c_x^2) c_{v,0}^2} \frac{\partial b_y}{\partial t} = 0.
\]  

(6)

In Eqs. (4) – (6), \( \tau_0 = k_B T_0 / m Q_0 \) is the characteristic heating time, \( C_{px} = C_{v,0} + k_B / m \) and \( C_{p0} = k_B \cdot (3\Omega_T - 3\Omega_0) / m \) are the high-frequency and low-frequency specific heats at constant pressure, respectively; \( C_{v0} = k_B \cdot 3\Omega T / m \) is the low-frequency specific heat at constant volume [9, 11]; \( c_x^2 = C_{px} k_B T_0 / m C_{v,0} \) and \( c_0^2 = C_{p0} k_B T_0 / m C_{v0} \) are squares of the high-frequency and low-frequency sound speeds, respectively; \( v_m = c^2 / 4\pi \sigma \) is the magnetic viscosity; \( \xi_0 = \tau_0 \rho_0 C_{v,0} (c_x^2 - c_a^2) / C_{v0} \) is the low-frequency bulk viscosity coefficient; \( c_a^2 = B_0^2 / 4\pi \rho_0 \) is the square of Alfvén speed. When isentropic instability criterion (2) is satisfied, the bulk viscosity coefficient becomes negative [11,13 – 15].

Equations (5), (6) have an analytical solution in the form of the weakly dissipating standing wave:

\[
v'_x = \varepsilon \cdot c_{f,0,x} \sin(\omega_n t) \sin(k_n x) \cdot \exp(-\alpha_{0,x} t),
\]

\[
\rho' = \varepsilon \cdot \rho_0 \cos(\omega_n t) \cos(k_n x) \cdot \exp(-\alpha_{0,x} t),
\]

(7)

\[
b_z = \varepsilon \cdot B_0 \cos(\omega_n t) \cos(k_n x) \cdot \exp(-\alpha_{0,x} t).
\]

In Eqs. (7), \( c_{f,0,x} = \sqrt{c_a^2 + c_x^2} \) are the phase speeds of fast magnetoacoustic wave; \( k_n = \pi n / l \) ( \( n = 1, 2, \ldots \) ), \( \alpha_{0,x} = c_{f,0,x} k_n \) are the eigenvalues and eigenfrequencies of a plasma layer; \( \alpha_0 = \xi_0 k_n^2 / 2\rho_0 + c_a^2 v_m k_n^2 / 2(c_x^2 + c_a^2) \) and \( \alpha_x = \xi_0 c_v^2 / 2\rho_0 (c_a^2 + c_x^2) c_{v,0}^2 + c_a^2 v_m k_n^2 / 2(c_x^2 + c_a^2) \) are low-frequency and high-frequency decrement (or increment if condition (2) is satisfied), respectively [14]; \( \varepsilon \sim 0 \) is the relative amplitude of perturbations in the magnetoacoustic wave. Solution (see Eq. (7)) is obtained in the assumption of weak amplification at the period, i.e. \( |\alpha_{0,x} / \omega_n| \sim 0 << 1 \) (further, indices of \( \alpha \) are neglected).

### 3.2 Alfvén wave

In order to take into account the nonlinear influence of the considered fast magnetoacoustic wave on the Alfvén wave propagated along the magnetic field, Equations (8) have been obtained from Eqs. (1) with using technique from [5, 6]. Equations (8) describe the nonlinear interaction of Alfvén waves with fast magnetoacoustic waves as well as Alfvén wave weak dissipation due to finite electric conductivity.

\[
\frac{\partial b_y}{\partial t} = (B_0 + b_z) \frac{\partial v'_y}{\partial z} - b_y \frac{\partial v'_x}{\partial x} + v_m \frac{\partial^2 b_y}{\partial x^2}; \quad \frac{\partial^2 b_y}{\partial t^2} = \frac{(b_z + B_0)}{4\pi} \frac{\partial b_y}{\partial z}.
\]  

(8)

Equations (8) can be joined in one Eq. (9) describing an evolution of the y-component of the magnetic field in the Alfvén wave:

\[
\frac{\partial^2 b_y}{\partial t^2} - \frac{2b_z}{B_0} \frac{\partial b_y}{\partial t} - B_0 v_m \left[ \frac{(\rho_0 - \rho') B_0^2}{4\pi \rho_0^2} \right] \frac{\partial^2 b_y}{\partial z^2} - v_m \frac{\partial^3 b_y}{\partial z^2 \partial t} = 0,
\]  

(9)
where $b_z, \tilde{b}_z$ are the first and second time derivatives of the perturbed magnetic field in the fast magnetoacoustic wave. Substitution of $b_y = h_y(z,t)\exp(b_z/B_0)$ and neglecting the higher-order terms in (9) leads to Eq. (10):

$$\frac{\partial^2 h_y(z,t)}{\partial t^2} - c_\omega^2 \frac{1 + b_z}{B_0} \frac{\partial^2 h_y(z,t)}{\partial z^2} - v_m \frac{\partial^3 h_y(z,t)}{\partial z^2 \partial t} = 0. \tag{10}$$

It can be seen from Eq. (10) that the fast magnetoacoustic wave changes weakly the Alfven-wave speed. Applying Fourier transform $h_y(z,t) = \int \tilde{h}_y(k_z,t)\exp(ik_zz)dk_z$ to Eq. (10) yields Eq. (11) describing the fast magnetoacoustic wave influence on the Alfven wave amplitude:

$$\frac{\partial^2 \tilde{h}_y(k_z,t)}{\partial t^2} + v_m k_z^2 \frac{\partial \tilde{h}_y(k_z,t)}{\partial t} + [\omega_n^2 + \delta \exp(-\alpha t)\cos(\omega_n t)] \tilde{h}_y(k_z,t). \tag{11}$$

In Eq. (11), $\delta = c_\omega^2 k_z^2 \cos(k_n x)$ in which x-coordinate plays a role of parameter. Without taking into account the finite-conductivity and heat-release, the obtained equation coincides with the Mathieu equation used in the analysis of the parametric interaction of the fast magnetoacoustic wave and the Alfven wave in [5].

Equation (11) can be solved in the resonant case $c_\omega k_z = \omega_n / 2$ with using of the method of multiple scales based on factorization of the time and unknown function ($t = t_0 + \theta t_1 + \theta^2 t_2...$ and $\tilde{h}_y = h_0 + \theta \tilde{h}_1 + \theta^2 \tilde{h}_2...$):

$$\tilde{h}_y(t) \approx \tilde{h}_{y0} \exp \left( -\frac{v_m k_z^2 t}{2} \right) \left[ \cosh(\Psi(t))\cos(\omega_n t) - \sinh(\Psi(t))\sin(\omega_n t) \right], \quad \Psi(t) = \frac{\delta(1-\exp(-\alpha t))}{2\alpha \omega_n}. \tag{12}$$

In the case of $v_m, \alpha = 0$, solution (see Eq. (12)) coincides with the solution obtained in [5]. Figure 1 illustrates how initially x-independent Alfven wave becomes x-dependent due to the parameter $\delta = c_\omega^2 k_z^2 \cos(k_n x)$. It can be seen from Eq. (12) that in case of the fully-conductive medium without the heat-release, there is the exponential growth of Alfven wave amplitude with the parametric increment $\delta/2\omega_n$ due to the nonlinear interaction with the fast magnetoacoustic wave. Finite-conductivity inclusion leads to the threshold condition of parametric amplification appearance $\delta > \omega_n v_m k_z^2$ and limitation of the time of parametric amplification existence $t < t_{ch}$, where $t_{ch} = (\phi + W(-\phi \exp(-\phi)))/\alpha, \ W$ is the Lambert function, $\phi = \delta / \omega_n v_m k_z^2$.

In the case of isentropically unstable heat-releasing media (where the increment $\alpha < 0$), there is the non-threshold parametric Alfven wave amplification for $t > t_{ch}$. In this case, the parametric increment grows exponentially with the time according to the function $\Psi(t)$. As a consequence, the Alfven wave amplitude grows bi-exponentially due to the parametric interaction with the unstable fast magnetoacoustic wave. The similar results for collinear geometry have been obtained in [3, 4].
4. Conclusion

In summary, inclusion of the finite conductivity results in the appearance of threshold condition and lifetime limitation of the Alfven wave parametric amplification. Heat-releasing inclusion leads to either additional damping of Alfven waves and decrease of parametric amplification lifetime or in the case of isentropic instability to the bi-exponential amplification of Alfven waves.

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