ANALYTICAL METHOD FOR TIME VARYING MESH STIFFNESS CALCULATION OF SPUR GEARS WITH GEAR ECCENTRICITY ERROR

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As one of the most common defects of gear systems, gear eccentricity error can influence the time varying mesh stiffness that is considered to be the major excitation source of gear systems. Accurate mesh stiffness under fault conditions is vital for dynamic simulation and further fault diagnosis, whereas most investigations focus on mesh stiffness calculation methods under tooth crack as well as other faults and ignore the effect of gear eccentricity error on mesh stiffness. To overcome this drawback, this paper presents an analytical method to calculate time varying mesh stiffness of spur gears with gear eccentricity error. Firstly, a novel gear mesh model that can determine the rotational basis of the gear system and actual tooth contact states is proposed. Then, an improved potential energy method is applied in the model for the mesh stiffness calculation. Finally, parametric studies are conducted to analyse the effect of gear eccentricity error on time varying mesh stiffness by numerical simulations. The results show that gear eccentricity error, leading to both amplitude modulation and frequency modulation, has significant impact on time varying mesh stiffness of spur gears, which is unallowable to be neglected. The proposed method improves the accuracy of time varying mesh stiffness with gear eccentricity error, and therefore contributes to dynamic vibration simulation and fault diagnosis of gear systems.

Keywords: spur gears, time varying mesh stiffness, gear eccentricity error, analytical method

1. Introduction

Gear transmission systems are one of the most common and important transmission mechanism and widely used in industry mechanical equipment to transmit torque and motion. As a major internal excitation source of gear systems, time varying mesh stiffness has great impact on the dynamic response and provides important operating condition information of gear systems [1], which significantly contributes to dynamic simulation and further fault diagnosis of gear systems. Accordingly, the calculation of the accurate time varying mesh stiffness has been a critical factor of gear dynamics.

Many studies have been carried out to calculate the mesh stiffness of healthy or fault gears. In literatures, analytical methods have been still regarded as a more promising method for gear mesh calculation because of the significant advantages in computation speed, modelling efficiency and satisfying results [2-3]. Yang and Lin [4] proposed an analytical model with considering tooth bending deflection, axial compression and Coulomb friction, and applied potential energy method to calculate gear mesh stiffness. Later, Tian [5] and Wu [6] refined Yang’s model [4] by adding tooth shear effect and derived the expressions of the mesh stiffness with three kinds of localized tooth faults (chipped, cracked and broken). Sainsot et al. [7] derived an analytical formula for gear body-induced tooth deflection and studied the fillet-foundation compliance, which can further improve the computational accuracy. Taking the flexibility between base circle and root circle into account, Wan et al.
[8] presented an improved calculation approach based on the previous works and analyzed the principle of gear root crack fault. Yu and Mechefske [9] introduced an analytical method to study corner contact effects on gear mesh stiffness and compared with two types of gear mesh stiffness model to stress its significance. Chaari et al. [10] investigated the effects of spalling widths and lengths or tooth breakage on mesh stiffness, and after that Ma et al. [11] and Jiang et al. [12] studied the effect of spalling defect on mesh stiffness.

From the above literatures, although the accuracy of gear mesh stiffness calculation has been improved, most studies focused on the tooth crack or spalling faults, and few researches considered the effect of gear eccentricity error on time varying mesh stiffness. The ignorance of gear eccentricity error can result in a great deviation in the calculation of mesh stiffness and may further lead to inaccurate understandings about dynamic simulation and fault diagnosis of gear systems.

To fill this gap, in this paper, an analytical method for mesh stiffness calculation with gear eccentricity error considered is presented. A novel gear mesh model considering gear eccentricity error that can determine the rotational basis and actual tooth contact states of the gear system is developed. Based on the proposed model, an improved potential energy method is applied for calculating the time varying mesh stiffness with gear eccentricity error. Finally, a parametric study is conducted to analyze the influence of gear eccentricity error on mesh stiffness.

2. Gear mesh model with gear eccentricity error

2.1 Determination of the rotational basis

In consideration of gear eccentricity error, the rotational center of the pinion is not coincident with the geometric center, and the center distance between the two geometric centers of a gear pair becomes variable correspondingly, as shown in Figure 1. Although the rotational centers are invariant, it is not calculation-friendly because most of gear parameters are based on the geometric center. Thus, to simplify further calculation and analysis, the geometric center is determined as the rotational basis.

![Figure 1: The rotational basis of the gear system considering gear eccentricity error.](image)

The actual pressure angle and center distance can be expressed as:

\[
\begin{align*}
\alpha_G &= \frac{a \cos \alpha_0}{a_G} \\
\alpha_G &= \arccos \left( \frac{a \cos \alpha_0}{a_G} \right) \\
a_G &= \sqrt{a^2 + \epsilon_e^2 - 2ae_e \cos \vartheta} \\
\end{align*}
\]

(1)

where \( a \) and \( \alpha_e \) are the theoretical center distance and pressure angle, and \( \epsilon_e \) is gear eccentricity error. The angular displacement \( \vartheta = \omega_1 t \) (\( \omega_1 \) is the angular velocity of the pinion) is associated with the rotational center, and according to the sine theorem, the rotational angle of the geometric center noted by \( \vartheta_G \) can be calculated as:
2.2 Tooth contact states of the gear system

With regard to an eccentric gear pair, the pitch circle and reference circle are not coincident and the action line direction is also back and forth, hence the initial angle and separate angle are proposed to determine tooth contact states of the gear system.

As illustrated in Figure 2, a pair of teeth is starting from contact point $A_1$ and separating from contact point $A_1'$. The action line $B_1B_2$ is tangent to the base circles of both pinion and gear at points $B_1$ and $B_2$. $C_{11}$ and $C_{21}$ are the middle points of tooth tip of the pinion and gear at the starting contact state, respectively, while $C_{11}'$ and $C_{21}'$ represent the similar points at the separating contact state. $F_{11}$ and $F_{11}'$ are the intersection points between tooth involute curve and base circle of pinion at the starting and separating contact states, respectively.

![Figure 2: Determinations of (a) initial angle and (b) separate angle.](image)

Based on the cosine theorem and involute function, the initial and separate angles are defined as:

\[
\chi_I = \angle B_iG_iA_i - \angle C_{1i}G_{i}A_{i1} = -\frac{\pi}{2N_1} - \text{inv}\alpha_0 + \tan \left[ \arccos \left( \frac{R_{g_{1i}}}{\sqrt{R_{s2}^2 + a_g^2 - 2R_{s2}a_g \cos \left( \arccos \frac{R_{g_{1i}}}{R_{s2} - \alpha_g} \right) }} \right) \right] \tag{3}
\]

\[
\chi_S = \angle B_iG_iF_{i1} - \angle C_{1i}G_{i}F_{i1} = \frac{\sqrt{R_{s1}^2 - R_{g_{1i}}^2}}{R_{s1}} - \theta_{b1} \tag{4}
\]

For a gear pair with specific contact ratio ($1 < \xi < 2$), there will be one or two pairs of teeth that engage simultaneously, which are so-called single tooth-mesh area and double tooth-mesh area, respectively. As shown in Figure 3, after defining the initial angle and separate angle (the two red lines are defined as the initial line and separate line correspondingly), it can be easily found that the number of tooth center lines (the blue dot dash lines) in between the initial line and separate line represents the number of pairs of mesh teeth. Similar to the definition of $C_{1i}$ described aforesaid, $C_{12}$ and $C_{13}$ are the points of the following two teeth of the pinion. Here, $\beta$ is defined as the angle between the first tooth center line $G_iC_{1i}$ and gear center line $G_O$, expressed by:
\[
\beta = \text{mod} \left( \frac{\theta_G}{2\pi} \frac{2\pi}{N_1} \right)
\]  

where \text{mod} means remainder, and \( N_1 \) is the tooth number of the pinion.

Taking Figure 3 (a) i.e., the case (a), as an example, the angle relation is

\[
\begin{align*}
\alpha_G + \beta - 2\pi/N_1 & \leq \chi_i \\
\alpha_G + \beta & \leq \chi_s
\end{align*}
\]

and it can be seen that only one pair of teeth is engaging, hence the operating pressure angle of the pinion is written as:

\[
\alpha_{m,1} = \alpha_G + \beta \quad \text{case (a)}
\]

Similarly, the operating pressure angles of the other three states can be denoted as:

\[
\begin{align*}
\alpha_{m,1} & = \alpha_G + \beta, \quad \alpha_{m,2} = \alpha_G + \beta - \frac{2\pi}{N_1}; & \text{case (b)} \\
\alpha_{m,1} & = \alpha_G + \beta - \frac{2\pi}{N_1}; & \text{case (c)} \\
\alpha_{m,1} & = \alpha_G + \beta - \frac{2\pi}{N_1}, \quad \alpha_{m,2} = \alpha_G + \beta - 4\pi/N_1; & \text{case (d)}
\end{align*}
\]

Note that the initial angle is time variant due to gear eccentricity error, and therefore, for a gear pair with specific parameters, case (a) and case (d) cannot exist at the same time and the selection of these two cases depends on the angle relation between \( \alpha_G, \beta, N_1 \) and \( \chi_i \) at the present moment.

Figure 3: Four tooth contact states with gear eccentricity error.

In Figure 3, \( \beta \) increases gradually as the pinion rotates, and finally the first tooth center line is beyond the separate line, i.e., the first pair of mesh teeth is separated, and when the second tooth center line exceeds the gear center line, it becomes the first tooth center and \( \beta \) is based on it correspondingly. \( \alpha_G \) is the actual pressure angle while the variable \( \alpha_m \) represents the so-called operating pressure angle which has different calculations under four tooth contact states (see Eqs (7) and (8)).
Figure 4: The operating pressure angles of a pair of mesh teeth.

Figure 4 shows the operating pressure angles of a pair of mesh teeth. According to geometry relation of triangles and characteristics of involute profile, the action line \( B_1 B_2 \) can be calculated as:

\[
B_1 B_2 = G_i O_j \sin \alpha_i = a_i \sin \alpha_i = B_1 F_1 + B_2 F_2 = R_{b1} (\alpha_{m1} + \theta_{b1}) + R_{b2} (\alpha_{m2} + \theta_{b2})
\]

where \( R_{b1} \) and \( R_{b2} \) are the base radii of the pinion and gear, \( \theta_{b1} \) and \( \theta_{b2} \) are the half tooth angles on the base circles of the pinion and gear, respectively. After simplification, the operating pressure angle of gear \( \alpha_{m2,i} \) will be:

\[
\alpha_{m2,i} = \frac{a_i \sin \alpha_i - R_{b1} (\alpha_{m1} + \theta_{b1})}{R_{b2}} - \theta_{b2} i = 1, 2
\]

3. Improved potential energy method for mesh stiffness calculation

In this paper, the calculation method for mesh stiffness is based on the commonly used potential energy method that is proposed by Yang and Lin [4] and further refined by Tian [5], Sainsot et al. [7] and Wan et al [8]. The total potential energy \( U_m \) stored in the mesh gear system is composed of Hertzian energy \( U_h \), bending energy \( U_b \), axial compressive energy \( U_a \), shear energy \( U_s \), and fillet-foundation energy \( U_f \), which can be obtained by:

\[
U_m = U_h + U_b + U_a + U_s + U_f = F^2 \left( \frac{1}{k_h} + \frac{1}{k_b} + \frac{1}{k_a} + \frac{1}{k_s} + \frac{1}{k_f} \right)
\]

where \( F \) is the acting force by the mesh tooth in contact point, \( k_h, k_b, k_a, k_s \) and \( k_f \) represent total, Hertzian, bending, shear, axial compressive, and fillet-foundation mesh stiffness, respectively.

Figure 5: Tooth model in the case of (a) \( R_b > R_f \) and (b) \( R_b \leq R_f \).
The general used potential energy method assumes the gear tooth as a variable cross-section cantilever beam fixed on the base circle, with the energy stored in the area between the base circle and root circle neglected [8]. Therefore, it is of great significance for further accurate mesh stiffness calculation to refine the potential energy method.

Figure 5 shows the tooth model in two cases. When \( R_b > R_f \), the bending, shear and axial compressive mesh stiffness can be calculated as [8]:

\[
\frac{1}{k_b} = \int_{-\alpha_0}^{\alpha} \frac{3\left(1 + \cos \alpha_m \left[\left(\theta_b - \alpha\right) \sin \alpha - \cos \alpha\right]\right)^2 \left(\theta_b - \alpha\right) \cos \alpha}{2EL \left[\sin \alpha + \left(\theta_b - \alpha\right) \cos \alpha\right]^3} \, d\alpha + \int_{r_{R_b-R_f}}^{R_{R_b-R_f}} \frac{3\left(1 - \cos \alpha_m \cos \theta_b \right) + x_i \cos \alpha_m}{2EL \left(R_b \sin \theta_b + R_c - \sqrt{R_c^2 - x_i^2}\right)} \, dx_i
\]

\[
\frac{1}{k_s} = \int_{-\alpha_0}^{\alpha} \frac{1.2(1+\nu)(\theta_b - \alpha) \cos \alpha \cos^2 \alpha_m}{EL \left[\sin \alpha + (\theta_b - \alpha) \cos \alpha\right]} \, d\alpha + \int_{r_{R_b-R_f}}^{R_{R_b-R_f}} \frac{1.2(1+\nu) \cos^2 \alpha_m}{EL \left(R_b \sin \theta_b + R_c - \sqrt{R_c^2 - x_i^2}\right)} \, dx_i
\]

\[
\frac{1}{k_a} = \int_{-\alpha_0}^{\alpha} \frac{(\theta_b - \alpha) \cos \alpha \sin^2 \alpha_m}{2EL \left[\sin \alpha + (\theta_b - \alpha) \cos \alpha\right]} \, d\alpha + \int_{r_{R_b-R_f}}^{R_{R_b-R_f}} \frac{\sin^2 \alpha_m}{2EL \left(R_b \sin \theta_b + R_c - \sqrt{R_c^2 - x_i^2}\right)} \, dx_i
\]

where \( E, L, \nu, r_c \) are Young’s modulus, tooth width, Poisson’s ratio, and root radius, respectively. \( \alpha, x_i, R_b, R_f \) are defined in Figure 5. Similarly, when \( R_b \leq R_f \), the bending, shear and axial compressive mesh stiffness can be calculated as [8]:

\[
\frac{1}{k_b} = \int_{-\alpha_0}^{\alpha} \frac{3\left(1 + \cos \alpha_m \left[\left(\theta_b - \alpha\right) \sin \alpha - \cos \alpha\right]\right)^2 \left(\theta_b - \alpha\right) \cos \alpha}{2EL \left[\sin \alpha + \left(\theta_b - \alpha\right) \cos \alpha\right]^3} \, d\alpha
\]

\[
\frac{1}{k_s} = \int_{-\alpha_0}^{\alpha} \frac{1.2(1+\nu)(\theta_b - \alpha) \cos \alpha \cos^2 \alpha_m}{EL \left[\sin \alpha + (\theta_b - \alpha) \cos \alpha\right]} \, d\alpha
\]

\[
\frac{1}{k_a} = \int_{-\alpha_0}^{\alpha} \frac{(\theta_b - \alpha) \cos \alpha \sin^2 \alpha_m}{2EL \left[\sin \alpha + (\theta_b - \alpha) \cos \alpha\right]} \, d\alpha
\]

where \( \tau = \tan(\arccos R_b/R_f) - \theta_b \) is defined in Figure 5 (b).

The Hertzian mesh stiffness [4] and fillet-foundation mesh stiffness [7] can be obtained by:

\[
k_b = \frac{\pi EL}{4(1-\nu^2)}; \quad \frac{1}{k_f} = \frac{f_f \cos^2 \alpha_m}{EL} \left[ L \left(\frac{u_f}{S_f}\right)^2 + M \left(\frac{u_f}{S_f}\right) + P \left(1 + Q \tan^2 \alpha_m\right)\right]
\]

In conclusion, the total effective mesh stiffness of single and double pair of mesh teeth, respectively, can be expressed as:

\[
k_{m\_single} = \frac{1}{k_h + k_{b1} + k_{b2} + k_{s1} + k_{s2} + k_{s1} + k_{s2} + k_{f1} + k_{f2}}
\]

\[
k_{m\_double} = \sum_{i=1}^{2} \frac{1}{k_{h,i} + k_{b,i} + k_{s,i} + k_{f,i}}
\]

where subscripts 1 and 2 denote the pinion and gear, respectively, and subscript \( i \) denotes the ordinal number of pairs of mesh teeth (\( i=1 \) for the first pair of mesh teeth and \( i=2 \) for the second pair).
4. Numerical calculation and results

Based on above formula derivations and analysis, in this section, the calculation procedure of the eccentric time varying mesh stiffness is presented and the effect of gear eccentricity error on time varying mesh stiffness is exhibited.

As can be seen from Eqs. (12) to (20), mesh stiffness is a function of the operating pressure angle that is variable and can be calculated under different tooth contact states with gear eccentricity error (see Eqs. (7), (8) and (10)). Consequently, the proposed calculation procedure can be expressed as:

a) Determine the geometric center as the rotational basis, and obtain the rotational angle;

b) Define the initial angle and separate angle (initial line and separate line correspondingly) and calculate the operating pressure angles of the pinion and gear;

c) Substitute the operating pressure angles in the mesh stiffness formulas calculated by the improved potential energy method to obtain the total effective time varying mesh stiffness.

In order to consider the effect of gear eccentricity error on time varying mesh stiffness, a numerical example is presented. The rotational frequency of the pinion \( f_{r1} \) is 50Hz. The basic parameters of the studied gear pair is shown in Table 1.

<table>
<thead>
<tr>
<th>Parameter (unit)</th>
<th>Pinion</th>
<th>Gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>26</td>
<td>40</td>
</tr>
<tr>
<td>Module (mm)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Pressure angle (deg)</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Teeth width (mm)</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Young’s modulus (N/mm²)</td>
<td>2.06E+11</td>
<td>2.06E+11</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.3</td>
<td>0.3</td>
</tr>
</tbody>
</table>

The effect of gear eccentricity on mesh stiffness is illustrated in Figure 6. Compared with Ref. [8] which does not consider the gear eccentricity effect, both amplitude and frequency modulations appear in this study. In Figure 6(b), when considering the effect of gear eccentricity, the sidebands \((nf_{m1} ± nf_{r1}, n \text{ is an integer})\) around mesh frequency and their harmonics \((nf_{m1})\) appear. Furthermore, as illustrated in Figure 6(a), considering the gear eccentricity value of the pinion varies from 0.2 mm to 0.5 mm incrementally and keeping other parameters unchanged, it can be seen that the periodic variations of amplitude are larger with the increase of gear eccentricity value.

Based on above analysis, gear eccentricity error has significant impact on mesh stiffness that is unallowable to be neglected, and with the proposed calculation method, the more accurate mesh stiffness can be computed and used for dynamic vibration simulation and fault diagnosis of gear systems.

![Figure 6: Effect of gear eccentricity on mesh stiffness in (a) time domain, and (b) frequency domain.](image-url)
5. Conclusions

This paper has demonstrated an analytical method to calculate time varying mesh stiffness of spur gears with gear eccentricity error considered. For the convenience of calculations, a novel gear mesh model is developed, which can determine the rotational basis, and then the initial angle and separate angle are proposed to determine actual tooth contact states of the gear system. Based on the model, an improved potential energy method is employed for the accuracy mesh stiffness calculation. Finally, the effect of gear eccentricity error on mesh stiffness is analysed through conducting a parametric study using numerical simulations. The results reveal that gear eccentricity error, leading to both amplitude modulation and frequency modulation, has greatly influenced the time varying mesh stiffness of gear systems, which is unallowable to be neglected. The proposed method contributes to gear mesh stiffness calculation and is useful for dynamic simulation and fault diagnosis of gear systems.

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