This paper concerns time harmonic wave motion in systems comprising beams with periodic translational spring attachments. Analytical solutions are developed and numerical results presented. The case of a single beam lying on periodic simple supports is well known and results are reviewed: the system exhibits a pass/stop band structure characteristic of periodic systems, with frequency bands where the propagation constants are pure real, pure imaginary or a complex conjugate pair. The case of two beams with periodic translational spring connections is then considered. This system is more complicated and exhibits rich dispersion behaviour. There are four pairs of waves, two pairs of which never propagate. The remaining two pairs exhibit pass/stop band behaviour. At low frequencies one pair propagates while the other is attenuating and is similar to waves in a beam on a continuous elastic foundation below the cut-off frequency. At higher frequencies locking and veering behaviour can be seen. For some values of the system parameters, the dynamics of the system can be approximated by that of a single beam on periodic simple supports. Furthermore, the response of each of the two periodically supported beams can be used to predict regions where all wavenumbers are purely imaginary for the coupled system, resulting in no propagation occurring.

Keywords: Periodic structure, veering, locking, beams

1. Introduction

The vibrational influence of cable harnesses attached to mechanical structures is of interest particularly in the aerospace, aviation and automobile industries [1, 2]. Previous studies have indicated that cable harnesses display beam-like behaviour and that the connections can be modeled as springs [3, 4, 5]. The double beam model is common for investigating the cable effects on beams [6, 4] where one beam represents the underlying mechanical structure, and the other, the cable harness. Previous work focused on modeling finitely long beam models using techniques such as FEM or SEM [6, 4]. This paper aims to apply periodic structure techniques to the problem to gain insight into of the behaviour of periodically coupled double beams. The system concerned shown in Fig. 1a consists of two infinitely long, Euler-Bernoulli beams, connected periodically by springs of stiffness $K$ at intervals of length $L$. Each beam has its own material properties; the beam stiffness $EI_{1,2}$ and mass per unit length $\rho A_{1,2}$. One possible option for a unit cell shown is in Fig. 1b.
2. Transfer Matrix Method for Calculating Propagation Constants

The procedure for determining the transfer matrix for the periodic cell of a double beam system is shown in Fig. 2. The state vector \( \phi_{a,b,L,R} \) comprises both the degrees of freedom and internal forces of both beams at the respective boundary. This can be represented as

\[
\phi_{a,b,L,R} = \begin{bmatrix} w_1 & \theta_1 & V_1 & M_1 & w_2 & \theta_2 & V_2 & M_2 \end{bmatrix}^T_{a,b,L,R}
\]

where \( w \) is the transverse displacement, \( \theta \) is the slope, \( V \) the internal shear force and \( M \) the bending moment. The transfer matrix \( T_S \) of a spring of stiffness \( K/2 \), relates \( \phi_L \) to \( \phi_a \) as well as \( \phi_b \) to \( \phi_R \), and the transfer matrix \( T_B \) relates \( \phi_a \) to \( \phi_b \). Thus, the state vectors at the left and right sides of the periodic cell represented in Fig. 1(b) can be written as

\[
\phi_R = T_S T_B T_S \phi_L = T \phi_L
\]

where \( T \) is the transfer matrix of the cell. \( T_B \) can be found by considering the relationship between forces and displacements in the section that is only governed by the motion of the beams, and \( T_S \) can be found by considering equilibrium and continuity conditions at the spring supports [7]. Periodicity conditions can be applied by requiring that the state vectors \( \phi_L \) and \( \phi_R \) are related by [8, 9, 10].

\[
\phi_R = e^{\mu} \phi_L
\]

where \( \mu \) is the propagation constant, or the Bloch wavenumber [11, 9, 10]. Substitution of Eq. (3) into Eq. (2) yields the eigenvalue problem [11, 12]

\[
T \phi_L = e^{\mu} \phi_L
\]

The eigenvalues of \( T \) are \( e^{\mu} \). The propagation constant \( \mu \) is generally complex with the real part \( \Re \{ \mu \} \) representing the attenuation across the cell and the imaginary part \( \Im \{ \mu \} \) representing the phase change across the cell. The solution give pairs of \( \pm \mu \) which represent waves in opposite directions. The eigenvectors of Eq. (4) give the wave modeshapes. There will be 4 pairs of solutions, two of which never propagate. The remaining two may form real pairs (evanescent waves), imaginary pairs (propagating waves) or complex pairs (attenuating waves).
3. A Single beam on Periodic Simple Supports

The results obtained by Mead [10] for a single beam on periodic simple supports can aid in the analysis of the double beam cases and are presented here. The uncoupled response (i.e. $K = 0$) is also given as a reference. In accordance with common practice, real parts of $\mu$ which correspond to attenuation are shown as positive, imaginary parts representing phase change are shown as negative $[11]$. In addition, given that waves exist in $\pm \mu$ pairs, and that only $\Im\{\mu\} \mod 2\pi$ is relevant, the phase change is shown only in the region $\{0, \pi\}$. Whilst there will be pairs of waves, one of which is propagating in the positive direction and the other in the negative direction, only one propagation constant is plotted for simplicity. The frequency parameter $\Omega = kL$ is used, where $k$ is the wavenumber in a bare beam. The response for the unsupported beam is shown in Fig. 3. It shows that there is one pair of evanescent waves and one pair of propagating waves. The behaviour shown for the propagating wave is typical of periodic systems because if $e^{i(kL)}$ is a solution, then $e^{\pm i(kL+2n\pi)}$ is also a solution. The region between the dashed lines $\{0, \pi\}$ successively corresponds to waves in the $\pm$ directions and are not the same waves. This region is referred to as the irreducible Brillouin zone $[8]$. For the simply supported case (Fig. 4), pass and stop band behaviour is present, where pass bands start when $\Omega = n\pi$ (the natural frequencies of the unit cell under free boundary conditions) and end at the natural frequencies of the unit cell under clamped conditions $[11]$.

![Figure 3: Single beam uncoupled propagation constants showing the first Brillouin zone](image)

![Figure 4: Propagation constants for single beam on periodic simple supports](image)

4. Numerical Results for Double Beam Systems

In this section, various results for the double beam system are presented and discussed.

4.1 Normalised variables

The material parameters of the double beam system can be rewritten in terms of the parameters

$$
\Omega = k_1 L; \quad \kappa = \frac{k_2}{k_1}; \quad \Lambda = \frac{EI_2 k_2^3}{EI_1 k_1^3}; \quad K_t = \frac{K}{EI_1 k_1^3}, \quad K
$$

(5)
where $\Omega$ is the frequency parameter, $\kappa$ the ratio of wavenumbers, $\Lambda$ the ratio of beam stiffness, $K_t$ the frequency dependent spring stiffness and in what follows, beam 1 is referred to as the primary beam.

### 4.2 Effect of Wavenumber Ratio on Propagation Constants of Uncoupled Beams

In the uncoupled case ($K_t = 0$), from the definition of $\kappa$ and $\Omega$ the propagating Bloch wavenumber of the primary beam repeats every $2\pi$, and thus for the secondary beam, repeats every $2\pi/\kappa$ as shown in Fig. 5. By the definition of $\Omega$, propagating wavenumbers of the uncoupled primary beam will always intersect the first Brillouin zone at $\Omega = n\pi$ intervals. The choice of which beam should be referred to as primary may be dependent on the specific application.

![Figure 5: Propagating wavenumbers in uncoupled beams](image)

#### 4.2.1 Bounding Points

The points where the imaginary part of the Bloch wavenumber meets the boundaries of first Brillouin zone (referred to as bounding points) of both beams also have an inherent periodicity. For rational $\kappa = a/b$ with $\{a, b\}$ being positive non zero integers with no common divisors, the periodicity is every $2b\pi$ as shown in Fig. 6 for $\kappa = 5/3$. Thus the Bloch waves for the entire frequency spectrum can be inferred by considering only the range between $0 < \Omega < 2b\pi$. Within this range, several possibilities for the locations of the bounding points can occur depending on the value of $\kappa$. One of the bounding points could lie far from another (Fig. 7a), bounding points could lie close together with similar phase or opposing phase (Fig. 7b), or bounding points could overlap exactly at the same frequency with the same phase or in antiphase (Fig. 7c). The values for $a$ and $b$ dictate which combinations will occur and at what frequencies. The bounding points at $\mu = 0$ of the primary beam denoted $P_n^{1+}$ occur every $2n\pi$. The bounding points at $\mu = \pi$, denoted $P_n^{1-}$ occur at $(2n - 1)\pi$. Similarly, bounding points for the secondary beam ($P_m^{2+}$ and $P_m^{2-}$) occur at $2\pi m/\kappa$ and $\pi(2m - 1)/\kappa$ respectively. Thus it can be seen that within the periodic range $2b\pi$ there are $b$ and $b - 1$ number of $P_n^{1-}$ and $P_n^{1+}$, $a$ and $a - 1$ number of $P_m^{2-}$ and $P_m^{2+}$ respectively. The pattern of the coinciding

![Figure 6: Repetition of propagating wavenumbers for $\kappa = 5/3$](image)
Figure 7: Possible orientations of bounding points showing (a) isolated bounding points, (b) in-phase and antiphase bounding points close-by and (c) in-phase and antiphase bounding points coinciding

bounding points shown in Fig. 7c can be predicted based on the parity of the values \( a \) and \( b \). When \( b \) is even, both antiphase and in-phase coinciding bounding points will be present. When \( b \) is odd and \( a \) is even, then the same statement holds. When \( b \) and \( a \) are both odd, then only in-phase coinciding bounding points will exist. These properties are of interest when considering the coupled system.

### 4.2.2 Intersection of Bloch Wavenumbers

The intersection of the Bloch wavenumbers are locations of interest because they are responsible for veering and locking of waves in the coupled system [13]. The three conditions that can occur are that the wavenumbers intersect and their slopes are of opposite sign (Fig. 7a for \( \Omega \) just less than \( 2\pi \)), they intersect with the slopes of the same sign (Figure 7a for \( \Omega \) just greater than \( 2\pi \)), or they intersect where two bounding points meet (Fig. 7c for \( \Omega = 2\pi \)). The latter has already been discussed above. The intersections are solutions to the equation [10]

\[
\cos(\Omega) = \cos(\kappa \Omega)
\] 

This yields two solutions

\[
\Omega_L = \frac{2\pi n}{\kappa + 1}; \quad \Omega_V = \frac{2\pi n}{1 - \kappa}
\] 

where the subscripts indicate the location of where locking (opposite slope) and veering (similar slope) will occur [13]. It is important to note that Eq. (7) will also give the solutions for where coinciding bounding points of the same phase intersect. Veering will not occur if the absolute difference between \( a \) and \( b \) is less than 3 when \( \kappa \) is rational.

### 4.3 Increasing Spring Stiffness

#### 4.3.1 Effect on Bounding Points

The effect of increasing spring stiffness on the bounding points of the Bloch wavenumbers is shown in Fig. 8. Branches which are always evanescent are not shown. It demonstrates how stop bands for wave groups start at the bounding points and grow in size towards the lower frequencies asymptotically approaching a maximum bandwidth. Single isolated bounding points will always grow stop bands (Fig. 9a). When the bounding points coincide and are in phase as shown in Fig. 9c, then only one of the branches will become a stop band. The specific wavenumber that develops a stop band is dependent on the stiffness ratio \( \Lambda \), where the branch associated with the less stiff beam will develop stop bands. When the bounding points are out of phase as shown in Fig. 9b, then both branches will develop stop bands each. Bounding points that are in phase and coincide can be thought of as a special case where one of the branches does not form a stop band but remains a propagating wave. For the case when two stop bands overlap, there are no Bloch waves that propagate through the structure. The
4.3.2 Veering and Locking of Imaginary Valued Propagation Constants

Propagation constants that intersect will result in veering or locking behaviour shown in Fig. 10. When the slopes of the uncoupled propagation constants are opposite in sign, then locking occurs (Fig. 10a). When the slopes are of the same sign, then veering will occur (Fig. 10b). Locking will result in the two imaginary valued propagation constants becoming complex conjugate pairs, resulting in a band where waves attenuate. The imaginary propagation constants that veer remain imaginary throughout the interaction. For locking, the Bloch waves are complex and thus propagate no energy through the structure resulting in a stop band.

4.4 Effect of Beam Stiffness Ratio

The beam stiffness ratio $\Lambda$ dictates which beam dominates in terms of behavior. $\Lambda < 1$ indicates that the primary beam is stiffer, whereas $\Lambda > 1$ indicates the secondary beam being stiffer. As a result of the normalisation of variables, a change in $\Lambda$ also implies a change in mass ratio $M_R = \Lambda \kappa$. It is important to note that changes in $\Lambda$ do not affect the number of stop bands, only their size. Values of $\Lambda << 1$ or $\Lambda >> 1$ indicate extreme situations (Figs. 11 and 12) where the stiffer beam is barely affected and the other demonstrates the typical response for a single beam on periodic simple supports (Fig. 4). The general behaviour for the stop bands is that increasing $\Lambda$ increases the bandwidth of the
Figure 10: (a) Bloch wavenumbers with opposite slopes locking to form a complex conjugate pair, and (b) Bloch wavenumbers with same slope veering

Figure 11: Double beam with dominant primary beam, $\kappa = 2; K_t = 100; \Lambda = 0.001$

stop band associated with the uncoupled primary beam and decreases the stop band associated with the secondary beam, decreasing $\Lambda$ yields opposite results (Fig. 13).

5. Concluding Remarks

It has been shown that some properties of the coupled double beam system can be predicted by an individual analysis of the uncoupled ($K_t = 0$) propagation constants, and the normalised parameters $\kappa$, $K_t$ and $\Lambda$. As such, general pass stop band behaviour can be inferred by observation of the uncoupled response. The wavenumber ratio $\kappa$ can be used to estimate the uncoupled response as well as the phases and locations of bounding points. The number of stop bands within a range of frequencies can be predicted by the intersection of the uncoupled propagation constants with themselves and the boundaries of the first Brillouin zone (bounding points). Increasing spring stiffness $K_t$ results in larger bandwidths of stop bands originating at bounding points and asymptotically approaching a maximum bandwidth. Increasing stiffness will result in propagation constants that intersect with opposite slope to lock, forming attenuation bands and those with the similar slope to veer and remain propagating. Observation of the beam stiffness ratio $\Lambda$ can infer which of the stop bands will have larger or smaller bandwidths and for very large or small values of $\Lambda$, determine the dominant behaviour of the system. For overlapping stop bands which develop from coinciding bounding points in anti-phase, there is strong indication that both individual stop bands have the same bandwidth, hence maximum bandwidth for no propagation, when the masses of the two beams are similar ($\Lambda \kappa = 1$). It should be noted that because the parameter for spring stiffness $K_t$ is dependent on frequency, the results demonstrated will differ from real double beam structures. There is however the benefit in that the behavior of the system can be generalised for ease of understanding, and if necessary, specific physical models can be implemented with frequency invariant stiffness.
REFERENCES


