A combined design method of impedance of acoustic nacelle is developed based on the Cremer impedance. First, the Cremer method is further extended to provide the theoretical optimum impedance for infinite uniform annular ducts in the presence of a single azimuthal mode and plug flow. Then, with the Cremer impedance being the initial value, the searching optimization based on a finite element sound propagation model is conducted to determine the impedance for maximizing noise reduction under the conditions of nearly actual duct geometry and flow. A design case of impedance of a typical aeroengine inlet is used to evaluate the method. The TL contour against impedance indicates that the combined design can avoid the local solution and find successfully the globally optimum impedance, thereby leading to the maximum sound attenuation. Additionally, the Cremer impedance is quite close to the final optimum impedance in this case, thereby providing not only a good initial impedance but also a proper searching space for the search optimization. Therefore, the combined design method is of high efficiency and accuracy. Keywords: Cremer method, optimum impedance, optimization design, acoustic treatment

1. Introduction

The latest ICAO noise level standard of chapter 14 stipulates a further reduction of at least 7 dB in the allowable cumulative noise level compared to the current version, which propels the further development of silent high bypass-ratio turbofan engines. One of the critical subjects is the forward propagating noise control, which is the destination of mounting acoustic liner in engine nacelle [1], thereby necessitating an efficient design method for the optimum impedance of acoustic treatment.

Currently, the following numerical optimization method, referred as the searching optimization method in this paper, is widely used [2, 3]: it exhaustively searches the optimal impedance of liner in an iterative process of maximizing the acoustic energy transmission loss (TL) within a lined duct. The method is suitable to design the silencer with complex geometry and flow, but it has two inevitable drawbacks. First, the sound field simulation in each iteration based on the sound propagation models [4–7] is quite time-consuming especially at high frequencies. Second, the improper initial impedance in search may lead to locally rather than globally optimal solution due to its sensitivity to the selection of initial value, therefore, the method sometimes fails to bring about maximum noise reduction.

On the contrary, the Cremer method is a purely theoretical design method of impedance, which can solve the optimal impedance for infinite uniform lined ducts for a certain azimuthal mode. In this
method, the branch point equations are derived from the eigen equation and its first \((j-1)\)-order partial differential equations, to solve the \(j\)-repeated eigenvalue (mode) and optimal impedance corresponding to the axial wavenumber with minimum imaginary part, which can result in the maximum sound attenuation. Chronologically, Cremer [8] first proposed the thought for rectangular ducts without flow. Then, Tester [9] extended it to circular ducts with uniform grazing flow, an asymptotic solution was obtained under a high frequency assumption. Recently, Kabral et al. [10] further derived the exact Cremer impedance available at lower Helmholtz numbers by removing the high frequency assumption. As for annular ducts, Zorumski and Mason [11] explored the corresponding Cremer impedance, but the branch point equations were not derived, and the grazing flow was not considered. Therefore, the complete branch point equations for annular ducts with a plug flow are derived in this paper. The most obvious advantage of the theoretical method is to give fast the optimal impedance and nearly maximum \(TL\), thereby being successfully applied in various silencer designs [12] recently.

From the elaborations above, the two design methods of impedance, the search optimization and the Cremer method, are found to be highly complementary. Therefore, a combined design method is developed in this paper. For an actual design problem, the geometry and flow are first assumed to be sectionally uniform, then, the Cremer theoretical optimum impedances for these segments are designed, whose mean value can be taken as the designed impedance if the acoustical treatment is expected to be uniform. Subsequently, with the Cremer impedance being the initial value, the searching optimization is conducted to determine the final optimum impedance for the actual problem. The combined method can fast select accurate initial impedance and appropriate searching range to conduct the acoustic design under complex geometry and flow, thereby being of high efficiency.

The combined design method is first illustrated in Section 2, the Cremer method for annular ducts lined on outer wall is elaborated in Section 2.1. A case is used to demonstrate its application in Section 3. The results are analyzed in Section 4. Finally, a summary is given in Section 5.

2. Description of the combined design method

2.1 Step 1: Cremer design method

2.1.1 Sound propagation within ducts

![Diagram of an annular flow duct lined on outer wall.](image-url)

Figure 1. Illustration of an annular flow duct lined on outer wall.

The procedure of combined design method developed in this paper consists of two steps, i.e. the Cremer design stage and the search optimization stage. In this step, the optimization problem of sound attenuation in an infinite uniform annular flow duct is first considered. The radius ratio \((RR)\) is defined as the ratio of inner radius \(a\) and outer radius \(b\), i.e. \(a/b\), as depicted in Fig. 1. As a particular case, the \(RR\) of circular duct is zero. A locally reacting liner with acoustic impedances \(Z_b\) is mounted on the outer wall to suppress the noise, and the inner wall is rigid. A cylindrical coordinate system is introduced with \(x\) axis being along the axial direction and \(r, \theta\) representing the radial and azimuthal directions, respectively, whose origin locates at the symmetry axis. A sound wave is incident from the infinite left, and a plug flow of Mach number \(M_a\), being parallel to the axial direction, passes through the duct. The objective is to achieve the optimum design of impedance \(Z_b\) to maximize the sound attenuation within a certain duct length.

For the sound field, the convective Helmholtz equation is employed as the governing equation:
\[
\left( ik + M_a \frac{\partial}{\partial x} \right)^2 p - \nabla^2 p = 0,
\]
where \( k = \omega/c \) is the free space wavenumber, and \( \omega \) and \( c \) represent the angular frequency and the speed of sound, respectively; \( M_a \) is given from the grazing flow velocity, i.e. \( V_a/c \), and \( V_a \) is positive/ negative when the flow has the same/ contrary direction as the incident sound wave. The soft-wall boundary condition can be given according to the Ingard and Myers’ work [13]:
\[
 ik \frac{\partial p}{\partial r} = \left( ik + M_a \frac{\partial}{\partial x} \right)^2 p/Z.
\]

For such an annular duct, the classical duct acoustics theory gives the solution to Eq. (1):
\[
p = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn}^+ J_m(k_{r, mn} r) + B_{mn} Y_m(k_{r, mn} r) e^{-im\theta} e^{-ik_{x, mn} x},
\]
where ‘\( mn \)’ denotes a mode with azimuthal order \( m \) and radial order \( n \), \( k_{r, mn}, k_{x, mn}^\pm \) and \( A_{mn}^\pm \) denote its radial and axial wavenumbers and complex modal amplitudes, respectively. The superscript ‘\( \pm \)’ denotes the positive and negative propagation modes, respectively. \( J_m \) and \( Y_m \) denote the \( m \)-th order Bessel function of the 1st and 2nd kind, respectively. \( B_{mn} \) are the coefficients determined by wall boundaries and sound mode. The two wavenumbers are related by the dispersion equation:
\[
k_{x, mn}^\pm = \frac{-M_a k^\pm \sqrt{k^2 - (1 - M_a^2) k_{r, mn}^2}}{1 - M_a^2},
\]

Theoretically, when a sound wave of a certain mode propagates through lined duct of length \( x \), its sound energy attenuation, i.e. transmission loss \( TL \), can be calculated from the imaginary part of \( k_x \):
\[
TL = 20 \log_{10} e^{-i m (k_x) x},
\]
where \( Im(k_x) \) should be a negative value for an axially decaying wave. An important rule is found that the theoretical \( TL \) grows as the \( Im(k_x) \) decreases. Therefore, the design in this step is to seek an optimum value of impedance \( Z_b \) to minimize the \( Im(k_x) \) for maximizing the sound attenuation.

### 2.1.2 Cremer method

The Cremer method aims at the 1st order radial mode, which is the most difficult to control for a certain azimuthal mode. For minimizing the \( Im(k_{r1}) \) and consequently minimizing the \( TL \), the strategy adopted in the method is, through degenerating the first two order radial modes into a single mode, to implicitly make the \( k_{r1} \) move away from the origin in the first quadrant, then elaborately select the merged \( k_{r1} \) with the largest module, which corresponds to the minimum \( Im(k_{r1}) \) according to the Eq. (4), the theoretical optimal impedance can be successively determined.

In this paper, the Cremer method for annular duct only lined on outer wall is first developed. First, substituting the acoustic field expression Eq. (3) of the acoustic mode \((m, n)\) into the wall boundary Eq. (2), the following equations are obtained for outer and inner walls, respectively:
\[
-Z_b k_r J'_m(k_r b) + B Y'_m(k_r b) = i k f^2 I_{m}(k_r b),
\]
\[
J'_m(k_r a) + B Y'_m(k_r a) = 0,
\]
where \( f_1 = 1 - M_a k_x/k \), and the subscript \( mn \) is omitted hereafter. On the inner rigid wall, the acoustic admittance, \( 1/Z \), is equal to zero, thus leading to the Eq. (7).

The coefficients \( B \) must be the same, so the eigen equation for such an annular duct can be obtained:
\[
D(k_r, Z_b) = 0: \quad \frac{-z_b}{ik} = \frac{f_1^2 [I_{m}(k_r b) Y'_m(k_r a) - Y_m(k_r b) J'_m(k_r a)]}{k_r [I_{m}(k_r b) Y'_m(k_r a) - Y_m(k_r b) J'_m(k_r a)]},
\]
which is denoted as \( D(k_r, Z_b) = 0 \) for brevity. The following variables is introduced:
\[
E_1 = k_r [J'_m(k_r b) Y'_m(k_r a) - Y_m(k_r b) J'_m(k_r a)],
\]
\[
E_2 = f_1^2 [I_{m}(k_r b) Y'_m(k_r a) - Y_m(k_r b) J'_m(k_r a)].
\]
Then, the eigen Eq. (8) is written compactly as below:

\[ D(k_r, Z_b) = 0: \]

\[ \frac{-z_b}{ik} = \frac{E_2}{E_1}, \quad (11) \]

where \( E_1 \neq 0 \) is required, and there are two undetermined quantities, i.e. radial eigenvalue \( k_r \), outer liner impedance \( Z_b \). For solving them, an additional constraint is necessary. Therefore, the first-order partial differential equation (PDE) of eigen equation with respect to \( k_r \), i.e. the branch point equation for annular ducts lined on outer wall, is directly derived and introduced:

\[ \frac{\partial D(k_r, Z_b)}{\partial k_r} = 0: \]

\[ \frac{E_1^2 E_1 - E_2 E_1^2}{E_1^2} = 0, \quad (12) \]

where the first-order partial derivatives of \( E_1 \) and \( E_2 \) are given:

\[ E_1' = \frac{E_1}{k_r} + bk_r[j_m'(k_r)b]Y_m'(k_r,a) - Y_m''(k_r,b)j_m'(k_r,a)] + ak_r[j_m'(k_r)b]Y_m''(k_r,a) - Y_m'(k_r,b)j_m'(k_r,a)] \]

\[ E_2' = \frac{2j_m' k_r}{r} + \frac{b j_m' k_r}{r} + a f_m^2[j_m'(k_r)b]Y_m'(k_r,a) - Y_m(k_r,b)j_m'(k_r,a)]. \]

The only unknown quantity left in the branch point Eq. (12), i.e. the radial eigenvalue \( k_r \), need to be solved first in Cremer method. In this article, the BFGS Quasi-Newton optimization method [14] is used to determine the optimum radial eigenvalue. Subsequently, the theoretical optimal impedance of the outer liner is algebraically solved from the Eq. (11).

Mathematically, the optimal \( k_r \) can be proved to be a double root. Once the impedance \( Z_b \) is determined, \( D(k_r, Z_b) \) reduces to \( D(k_r) \), and its Taylor expansion near an arbitrary eigenvalue \( k_{rn} \) is:

\[ D(k_r) = D(k_{rn}) + D'(k_{rn})(k_r - k_{rn}) + \frac{D''(k_{rn})}{2!}(k_r - k_{rn})^2 + \cdots = 0, \quad (15) \]

where \( D(k_{rn}) \) should be equal to zero. Under the constraint of Eq. (12), i.e. \( D'(k_{rn}) = 0 \), the Eq. (15) reduces to the following form after ignoring the higher-order terms:

\[ D(k_r) = \frac{D''(k_{rn})}{2!}(k_r - k_{rn})^2 = 0. \]

Therefore, the eigenvalue \( k_{rn} \) is a double eigenvalue of the eigen equation, which corresponds to a merging mode of first two order radial modes, physically.

### 2.2 Step 2: searching optimization method

Because the geometry and flow are simplified and only the dissipative TL is optimized, the Cremer theoretical optimum impedance should be somehow different from the final optimum impedance. Therefore, in order to consider actual duct geometries, flow and total transmission loss, a sound propagation model based on the discontinuous Galerkin finite element method (FEM) is adopted in the searching optimization stage. An exhaustive search of the optimum impedance is performed, simultaneously, the sound propagation problem is solved by the model in each impedance iteration. The iterative search continues until the fraction of incident sound energy transmitting through the lined duct is numerically minimized. It needs to be mentioned that the simulation of physical field and the impedance search in this step are implemented in the commercial software COMSOL [15].

In acoustic field simulation, when the impedance and incident sound energy \( W_{in} \) are given, the transmitted sound energy \( W_{tr} \) in the downstream of lined section can be numerically obtained. Then, the total transmission loss, taken as the objective function needing to be maximized, can be calculated:

\[ TL = 10 \log_{10} \frac{|W_{in}|}{|W_{tr}|}. \]

The search of impedance is usually limited in a certain impedance space by giving the upper and lower limits of the acoustic resistance \( Z_r \) and acoustic reactance \( Z_i \) respectively:

\[ \{ (Z_{br}, Z_{bi}) | Z_{br} \in [Z_{br,min}, Z_{br,max}], Z_{bi} \in [Z_{bi,min}, Z_{bi,max}] \}. \]
In this step, the Nelder-Mead optimization method [16] is utilized to search the optimal solution. Additionally, due to the sensitivity of search to the initial impedance, the improper selection of initial value usually consumes much more calculation time especially at high frequencies, and may result in locally optimum impedance for some optimization algorithms. Therefore, the corresponding Cremer impedance is taken as the initial value of optimization to avoid the blindness of selection, and a more appropriate searching range can be selected to reduce the time cost. Through combining the two methods, the optimum impedance for actual design problem can be exactly and fast determined.

3. Application of the combined design method

A typical axisymmetric aeroengine inlet used by Rienstra [4], whose RR of annular cross section varies from 0.4236 down to 0, is used to illustrate the design method. The red lines in Fig. 2 show the geometry of inlet, whose inner and outer radii of the flow channel are given in the formulae (6) and (7) of [4]. The outer wall of inlet is lined by locally reacting liner of impedance $Z_b$, whose axial length is 1.86393, while the surface of hub is rigid. The same non-dimensionalization as theirs is used.

The two ends of the flow channel are extended to sufficiently remote positions, where the flow is set to be uniform and axial, and the outer sides of the flow channel are rigid walls other than the lined wall. The forward propagating noise of mode (10, 1) and $\omega=16$ emits from the source plane, where the mean flow $M_a$ is -0.5 and both the mean density and sound speed are 1. The sound is forced to be reflective at the two ends, which will be introduced later. The sound propagation model within such an infinite duct is set up, the task is to design the impedance of liner to maximally suppress the noise.

![Figure 2. Geometry of a typical aeroengine inlet and the corresponding computational domain in simulations.](image)

![Figure 3. (a) The mean axial $M_a$ and (b) the Cremer impedance on the selected cross sections.](image)
obtained from the simulated mean flow, whose calculation will be demonstrated in Section 3.2, and plotted in Fig. 3a. The densities and sound speeds in these segments are assumed to be identical to those on the source plane. Each segment is regarded as part of an infinite uniform lined duct, whose inner and outer radii and $M_a$ of assumed uniform flow are identical to those of corresponding central cross section. Subsequently, the Cremer method in Section 2.1 is utilized to design the Cremer impedance for each duct, which is ideally regarded as the optimized impedance of corresponding segment, illustrated in Fig. 3b. The Cremer impedance of circular duct can be obtained through calculating that of annular duct with the same outer radius and the RR tending zero. Then, the Cremer impedance of acoustically uniform liner is obtain from the average value of these optimized impedances, i.e. $Z_b^{\text{cre}}=3.0515-1.2926i$, where the superscript ‘Cre’ denotes the result designed by Cremer method.

### 3.2 Searching impedance

The numerical model is built to simulate the physical field in every iteration, whose computational domain is depicted in Fig. 2. The mean flow field is first simulated. The normal velocity boundary is imposed on the right terminal plane, and the aerodynamic parameters on the source plane are constrained by the given values. The simulation of the mean background flow is implemented by the compressible potential flow model in COMSOL. Subsequently, the perfectly matched layers are added at the two ends of domain. The sound field can be obtained under the designated incident sound mode by solving the linearized potential flow equation in COMSOL. Based on the sound propagation model, the iterative search for optimization is initialized with $Z_b^{\text{cre}}$. For analysing the design optimality, a relatively large searching space $\{(Z_{br}, Z_{bi}) | Z_{br} \in [-0.1, 8], Z_{bi} \in [-8, 8]\}$ is selected. In practical design, the search can be limited in the proper proximity of Cremer impedance for saving time. The final optimum impedance for actual geometry and flow is obtained, i.e. $Z_b^{\text{com}}=3.1933-1.2013i$, where the superscript ‘Com’ denotes the result designed by the combined method.

### 4. Result and analysis

#### 4.1 Validation of the propagation model

First, the case [4] of $m=10$, $\omega=16$, $M_a=-0.5$ and $Z_b=2$-1i is selected as the validated example for our sound propagation model. For such a case, the total TL calculated by our model is $27.9\, \text{dB}$, against the $\text{TL}$ of $27.2\, \text{dB}$ by FEM in their paper. Additionally, the simulated contour of sound pressure amplitude, normalized by the maximum amplitude, is exhibited in Fig. 4a, which can be compared with the corresponding contour in their Fig. 6(d). The comparison indicates that the two contours are almost consistent. Therefore, the sound propagation model employed in our optimization is accurate enough in terms of the $\text{TL}$ and the sound field contour.

#### 4.2 Assessment of designed impedance

The $\text{TL}$ of the two designed impedances, i.e. $Z_b^{\text{cre}}$ and $Z_b^{\text{com}}$, are calculated by means of the sound propagation model under the given conditions, respectively, and listed in Table 1 together with that of the case of $Z_b=2$-1i. The results show that the designed impedances bring about evident $\text{TL}$ increments when compared with the impedance of 2-1i. Their contours of normalized sound pressure amplitude, presented in Fig. 4b and 4c, differ distinctly from that of $Z_b=2$-1i, which shows that the optimized liners absorb a majority of sound energy at the leading end, while the common liner of $Z_b=2$-1i almost needs twice time the length of the former to realize the identical sound attenuation.

The Cremer impedance is quite close to the impedance designed by the combined method, although the Cremer optimized $\text{TL}$ is $19\, \text{dB}$ lower than the maximum $\text{TL}$. This can be made clearer, when the contour of $\text{TL}$ against the impedance in the whole searching space is plotted in Fig. 5. The gradient of $\text{TL}$ against impedance around the optimum solution is very large, which explains why such a small difference of impedance leads to such a large $\text{TL}$ increment. Nevertheless, in consideration of the steep slopes there, the $\text{TL}$ increment may not be reflected completely in practice due to the deviations of working conditions, impedance model and liner manufacture. Therefore, the Cremer
Figure 4. The contours of normalized sound pressure amplitude when $m=10$, $\omega=16$, $M_a=-0.5$: (a) $Z_b=2-1i$, (b) $Z_b^{Cre}=3.0515-1.2926i$ and (c) $Z_b^{Com}=3.1933-1.2013i$.

Table 1. The designed impedances and corresponding $TL$ when $m=10$, $\omega=16$ and $M_a=-0.5$.

<table>
<thead>
<tr>
<th>Category</th>
<th>Impedance</th>
<th>TL (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compared case</td>
<td>2-1i</td>
<td>27.9</td>
</tr>
<tr>
<td>Designed by Cremer method</td>
<td>3.0515-1.2926i</td>
<td>53.4</td>
</tr>
<tr>
<td>Designed by Combined method</td>
<td>3.1933-1.2013i</td>
<td>72.4</td>
</tr>
</tbody>
</table>

Figure 5. The contour of $TL$ against impedance when $m=10$, $\omega=16$, $M_a=-0.5$.

Theoretical optimum impedance can be regarded as a successful design in the engineering for this case. Additionally, there is a locally optimum solution of impedance being about $4.8-1.2i$, whose $TL$ of $52.6 \, dB$ is less than the two optimized $TL$. Therefore, the combined design avoids the local solution and finds successfully the globally optimum impedance, thereby maximizing the sound attenuation.

In this case, for evaluating the optimality, a fairly large searching space is used. Actually, the results indicate that the search can be limited appropriately in the proximity of the Cremer impedance, which can lead to a smaller searching impedance space, thereby further increasing the efficiency.
5. Conclusions

In this article, the Cremer method for annular ducts lined on outer wall is first derived. Subsequently, a combined design method of acoustic impedance for annular aeroengine nacelle is developed based on the Cremer impedance and the searching optimization. The method is applied to design the optimum impedance of a typical aeroengine inlet. The TL contour against impedance indicates that the combined optimization design avoids the local solution and finds successfully the globally optimum impedance, thereby bringing about the maximum noise suppression for this application. Additionally, the Cremer theoretical optimum impedance is quite close to the final optimum impedance in this case, thus providing not only a good initial impedance but also a proper searching space for the search optimization. Therefore, the combined method is of high efficiency and accuracy.

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REFERENCES