ISOGEOMETRIC ANALYSIS FOR THE WAVE PROPAGATION IN A NON-UNIFORM DUCT

Yaqiang Xue, Guoyong Jin*
College of Power and Energy Engineering, Harbin Engineering University, Harbin, 150001, China
*corresponding author, email: guoyongjin@hrbeu.edu.cn

Hui Shi
Marin design & Research Institute of China, Shanghai, 200011, China

This paper investigates the performance of isogeometric finite element method for solving wave propagation in a non-uniform duct with rigid walls. The duct domain and Helmholtz equation are parameterized by Non-Uniform Rational B-splines (NURBS) basic functions, which have high-order continuous derivatives and simplify the mesh generation. The accuracy of the current method are testified by comparison with the analytical solution. Numerical results present that the pollution error is considerably reduced by increasing the order of the basic functions. Furthermore, the influence of the geometric parameter on wave propagation is discussed in detail.

Keywords: Isogeometric analysis, Duct, Helmholtz equation, Pollution error

1. Introduction

Wave motion is one of a range of physical phenomena and it can be characterized by the Helmholtz equation. Finite element method (FEM) [1] and boundary element method (BEM) [2] are two numerical techniques which can simulate the phenomenon. However, when traditional FEM is used, approximability requirements are severe and the numerical solution suffers from the pollution error [3] in the high frequency. Some techniques were developed to reduce the pollution error, including the partition of unity method (PUM) proposed by Melenk and Babuska [4] and the ultraweak variational formulation (UWVF) by Cessenat and Despres [5] and the least-squares method (LSM) by Monk and Wang [6].

Isogeometric analysis (IGA) is a novel numerical method proposed by Hughes and coworkers [7, 8] to unify CAD and CAE. In this case, the NURBS functions are employed in geometric construction together with finite element approximation. Compared with the Lagrange and Hermite functions in the FEM, the NURBS functions can maintain exact geometry. The IGA has been successfully utilized to solve a range of practical problems [9]. In acoustic problems IGA has been applied and attained excellent results. Simpson et al. analyzed interior and exterior acoustic problems [10] by an isogeometric boundary element method based on T-splines. Wu et al. [11] investigated the superiorities of IGA over $C^0$-FEM for the interior acoustics problems. Khajah et al. [12] presented an isogeometric analysis of time-harmonic exterior acoustic problems. Coox et al. [13] studied the performance of IGA for solving stationary acoustic problems in two dimensions and showed that the IGA exhibits higher accuracy than the traditional FEM, they [14] also solved three-dimensional acoustic problems in the frequency domain using IGA in conjunction with an indirect Boundary Element Method (BEM). Peake et al. [15] studied two-dimensional Helmholtz problems in the framework of Extended isogeometric BEM (XIBEM). Nørtoft et al. [16] presented a flow-acoustic model of the propagation of
sound through a slowly moving fluid in two-dimensional ducts by implementing IGA and examined how the acoustic signal varies with sound frequency, flow speed and duct geometry. Recently, Dinachandra and Sethuraman [17] investigated the modeling and analysis of the dynamic behavior of acoustic fluid-structure interaction problems based on IGA, they [18] also proposed a plane wave enriched Partition of Unity Isogeometric Analysis (PUIGA) for solving two-dimensional Helmholtz problems in acoustics.

In this paper, an isogeometric approach is applied for solving Helmholtz problem in two dimensions and the effects of geometric parameter on wave propagation in a non-uniform duct are investigated.

2. Formulation of IGA for Helmholtz equation

2.1 B-splines and NURBS basic functions

A set of non-negative real numbers are arranged in the knot vector \( E = \{ \xi_1, \xi_2, \ldots, \xi_{n+p+1} \} \), where \( n \) and \( p \) separately denotes the number and polynomial order of basic functions, respectively. The basic functions of degree \( p \) exhibit \( p-k \) order continuous derivatives at a single knot, where \( k \) denotes the multiplicity of the knot. Next, the B-spline basic function \( N_{i,p}(\xi) \) is defined recursively by the Cox-de-Boor formula [19]

\[
N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi \leq \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } p = 0 \tag{1}
\]

\[
N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad \text{for } p \geq 1 \tag{2}
\]

According to the tensor product of two univariate B-spline basic, the bivariate NURBS basic functions can be defined as:

\[
R_{i,j}^{p,q}(\xi, \eta) = \frac{w_{i,j} N_{i,p}(\xi) M_{j,q}(\eta)}{\sum_{i=1}^{n} \sum_{j=1}^{m} w_{i,j} N_{i,p}(\xi) M_{j,q}(\eta)} \tag{3}
\]

where \( w_{i,j} \) is 2D NURBS weight, the B-spline basic function \( M_{j,q}(\eta) \) bears order \( q \) in the \( \eta \) dimension. The NURBS basic functions are degenerated into B-Spine basis functions if \( w_{i,j} = 1 \). Figure 1 depicts the one-dimensional quadratic and two-dimensional B-Spine basis functions. A NURBS surface can be constructed by control points \( B_{i,j} \) and the NURBS basic functions

\[
S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) B_{i,j} \tag{4}
\]

2.2 Description of the model

The non-uniform duct model is constructed by two knot vectors and four control points which are given in Table 1 and Table 2, respectively. We assume that the lower and upper walls are perfectly rigid and denote the angle between upper wall \( \Gamma_3 \) and \( \gamma = 2 \) with \( \theta \) (Fig. 2). Wave propagates in this
duct and the goal is to find acoustic pressure $u$ which satisfies the Helmholtz equation and boundary conditions as follows:

$$\nabla^2 u + k^2 u = 0 \quad \text{in } \Omega$$  \hspace{1cm} (5.1)

$$\frac{\partial u}{\partial n} = \cos(mx y) \quad \text{on } \Gamma_1: x = 0$$  \hspace{1cm} (5.2)

$$\frac{\partial u}{\partial n} + i ku = 0 \quad \text{on } \Gamma_2: x = 2$$  \hspace{1cm} (5.3)

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_3, \Gamma_4$$  \hspace{1cm} (5.4)

where $k$ is the wave number, $n$ is the unit outward normal vector on $\partial \Omega$ and $m$ is the mode number. The inlet boundary $x = 0$ has an inhomogeneous Neumann condition, and an absorbing boundary condition is applied on the outlet boundary $x = 2$. It is noted that the outlet size is influenced by the parameter $\theta$. Besides $\theta > 0$ means that the outlet size is larger than the inlet one and $\theta < 0$ indicates that the outlet size is smaller than the inlet one.

Table 1: Polynomial degrees and knot vectors for the duct.

<table>
<thead>
<tr>
<th>Direction</th>
<th>Degree</th>
<th>Knot vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>$p = 1$</td>
<td>$E = {0, 0, 1, 1}$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$q = 1$</td>
<td>$H = {0, 0, 1, 1}$</td>
</tr>
</tbody>
</table>

Table 2: Control points for the duct.

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>$B_{ij}$</th>
<th>$w_{ij}$</th>
<th>$(i, j)$</th>
<th>$B_{ij}$</th>
<th>$w_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>(0, 0)</td>
<td>1</td>
<td>(1, 2)</td>
<td>(2, $2\tan(-\theta)$)</td>
<td>1</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>(0, 1)</td>
<td>1</td>
<td>(2, 2)</td>
<td>(2, $2+2\tan\theta$)</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 2: Geometry and coordinate system of a non-uniform two-dimensional duct.

### 2.3 Isogeometric formulation

The weak form of the Helmholtz equation can be derived by multiplying a weight function $v$, employing Green formula and applying the prescribed boundary conditions as

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega - k^2 \int_{\Omega} uv d\Omega + ik \int_{\Gamma_1} uv d\Gamma - \int_{\Gamma_2} \cos(mx y)vd\Gamma = 0$$  \hspace{1cm} (6)

The key idea of IGA is that the NURBS basic functions which are utilized to exactly build the original model will describe the solution space at the same time. Thus, the coordinate of physical point and the acoustic field of duct can be approximated as

$$x^h(\xi, \eta) = \sum_{A=1}^{N} R_A(\xi, \eta)x_A \quad \text{and} \quad u^h(\xi, \eta) = \sum_{A=1}^{N} R_A(\xi, \eta)u_A$$  \hspace{1cm} (7)
where \( R_A(\zeta, \eta) \) is the basic function, \( N \) is the total number of control point and \( \mathbf{u}_A \) designates the acoustic pressure vector with respect to the control point \( \mathbf{x}_A = (x_A, y_A)^T \).

In the context of Galerkin method, both the shape functions and weight functions have the same functional space. Substituting Eq. (7) into Eq. (6), the Galerkin weak formulation for the acoustic problem is derived as

\[
\left( \mathbf{K} - k^2 \mathbf{M} - ik \mathbf{C} \right) \mathbf{U} = \mathbf{F}
\]

in which \( \mathbf{K}, \mathbf{M}, \mathbf{C} \) and \( \mathbf{F} \) are the global stiffness, mass, damping matrices and force vector, respectively, given by

\[
\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{B} \, d\Omega \quad \text{where} \quad \mathbf{B}_A = \left\{ R_{A,x}, R_{A,y} \right\}^T
\]

\[
\mathbf{M} = \int_{\Omega} \mathbf{R}' \mathbf{R} \, d\Omega
\]

\[
\mathbf{C} = \int_{\Gamma} \mathbf{R}' \mathbf{R} \, d\Gamma
\]

\[
\mathbf{F} = \int_{\Gamma_1} \cos(m \pi y) \mathbf{R}' \, d\Gamma
\]

### 3. Results and discussions

In this section, several selected numerical examples are studied to demonstrate the accuracy of the current method. Furthermore, the effects of the geometric parameter on wave propagation in a duct are discussed.

#### 3.1 Convergence study and validation

In the case of \( \theta = 0^\circ \), the outlet size is equal to the inlet one, namely the duct is uniform, and the analytical solution [20] to the Helmholtz problem is expressed as

\[
u^{\text{ex}}(x, y) = \cos(m \pi y)(A_1 e^{-i k x} + A_2 e^{i k x})
\]

in which \( k = \sqrt{k^2 - (m \pi)^2} \) and coefficients \( A_1 \) and \( A_2 \) fulfill the equation

\[
\begin{pmatrix}
k_x \\
- k_x \\
(k - k_x) e^{-2i k_x} \\
(k + k_x) e^{2i k_x}
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix}
= \begin{pmatrix}
-i \\
0
\end{pmatrix}
\]

The solution represents propagating modes if the parameter \( m < k / \pi \) and evanescent modes if \( m > k / \pi \). Only the propagating modes are studied in this section. IGA is an element-based approach developed from traditional FEM, and we define the density of discretization as the number of elements per wavelength \( \lambda \) and designate it with \( n_\lambda \). Besides \( N_d \) and \( M_d \) which are the numbers of elements along two directions can be obtained by using the MATLAB inbuilt function \textit{ceil}

\[
N_d = \left\lfloor \frac{n_\lambda k}{2 \pi} \right\rfloor, \quad M_d = \left\lfloor \frac{2n_\lambda k}{2 \pi} \right\rfloor
\]

And the following relative error is defined to measure the accuracy of numerical solution

\[
\varepsilon_{\text{err}} = \left\| \nu^{\text{ex}} - \nu^b \right\| L^2(\Omega) / \left\| \nu^{\text{ex}} \right\| L^2(\Omega) \quad \text{where} \quad \left\| \nu^{\text{ex}} \right\| L^2(\Omega) = \left( \int_{\Omega} \left| \nu^{\text{ex}} \right|^2 \, d\Omega \right)^{1/2}
\]

Table 3. Variation of relative error \( \varepsilon_{\text{err}} \) with different degree of NURBS and density of discretization \( n_\lambda \) for wave number \( k = 20 \) and mode number \( m = 6 \).

<table>
<thead>
<tr>
<th>( p = q )</th>
<th>( n_\lambda = 4 )</th>
<th>( n_\lambda = 6 )</th>
<th>( n_\lambda = 8 )</th>
<th>( n_\lambda = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5172956</td>
<td>0.1212383</td>
<td>0.0439526</td>
<td>0.0193570</td>
</tr>
<tr>
<td>3</td>
<td>0.0561990</td>
<td>0.0036881</td>
<td>0.0008540</td>
<td>0.0003029</td>
</tr>
<tr>
<td>4</td>
<td>0.0065647</td>
<td>0.0003480</td>
<td>0.0000801</td>
<td>0.0000264</td>
</tr>
</tbody>
</table>
The numerical results are computed for the wave number \( k = 20 \) and the corresponding highest propagating mode number \( m = 6 \). Table 3 shows the variation of \( \varepsilon_{err} \) with different degree of NURBS and density of discretization \( n_\lambda \). As can be observed, the calculational results show excellent convergence while gradually increasing the number of element meshes. The variation of \( \varepsilon_{err} \) with different wave number and degree of NURBS is shown in Fig. 3. Furthermore, the pollution error is nearly negligible, especially for higher-order basic function such as \( p = 3 \) and \( p = 4 \). The discretization density \( n_\lambda = 10 \) and cubic NURBS element can obtain relatively accurate results, which are better than quadratic element and have less computational cost than quartic element. Hence, this mesh will used in following study of the paper. The real parts of exact solution and IGA solution are shown in Fig. 4(a) and Fig. 4(b), respectively. The corresponding absolute error \( |u_{ex} - u_h^k| \) is depicted in Fig. 4(c). We clearly find that the absolute error is the order of \( 10^{-5} \).

The effects of angle \( \theta \)

Consider \( \theta = \arctan(-0.05) \), namely the width of the outlet is 1.8. The real parts of the IGA solutions for wave number \( k = 20 \) and mode number \( m = 5 \) are plotted in Fig. 5. It can be seen from Fig. 5 that \( m = 5 \) is the highest propagating mode number and \( m = 6 \) is the lowest evanescent mode. This
phenomenon shows that when the outlet width reduces from 2 to 1.8 the highest propagating mode number decreases.

![Figure 5: The real parts of the IGA solutions for $k = 20, \theta = \arctan(-0.05)$ (a) $m = 5$ (b) $m = 6$.](image)

Then we take $\theta = \arctan0.05$ and the corresponding outlet width is 2.2. The real parts of the IGA solutions for wave number $k = 20$ and mode number $m = 6, 7$ are shown in Fig. 6. It can be seen that $m = 6$ is the highest propagating mode number and $m = 7$ is the lowest evanescent mode. The highest propagating mode number remains the same when the outlet width enlarges from 2 to 2.2.

![Figure 6: The real parts of the IGA solutions for $k = 20, \theta = \arctan0.05$ (a) $m = 6$ (b) $m = 7$.](image)

Furthermore, the influence of parameter $\theta$ on the highest propagating mode number of the duct is investigated in detail. The numerical results are computed for the wave number $k = 20, 40, 80$ and the evolution of the corresponding highest propagating mode number $m_{\text{max}}$ with parameter $\theta$ is tabulated in Table 4. It is obvious that for a certain wave number $k$ the parameter $m_{\text{max}}$ increases by increasing the angle $\theta$ in the range of $\theta < 0$ and $m_{\text{max}}$ is unchanged in the range of $\theta > 0$.

<table>
<thead>
<tr>
<th>Wave number</th>
<th>The high propagation mode number $m_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 20$</td>
<td>$\theta = \arctan(-0.1)$ 4  $\theta = \arctan(-0.05)$ 5  $\theta = 0$ 6  $\theta = \arctan0.05$ 6  $\theta = \arctan0.1$ 6</td>
</tr>
<tr>
<td>$k = 40$</td>
<td>8</td>
</tr>
<tr>
<td>$k = 80$</td>
<td>20</td>
</tr>
</tbody>
</table>

4. Conclusions

In this paper, isogeometric analysis method is applied for solving Helmholtz problem in two dimensions. The non-uniform duct model and its interior acoustic pressure are described by Non-Uniform Rational B-Splines (NURBS) basic functions. When the lower and upper walls of the duct are parallel to each other, the accuracy and convergence of the current method are investigated by comparison with the analytical solution. The pollution error is clearly alleviated on the base of high-order continuous basic functions. The effects of the geometric parameter on the highest propagating mode number are discussed as well.
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