ACOUSTIC METAMATERIALS APPLIED ON CIRCULAR INTERFACES

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A number of planar-interface types of acoustic metamaterials (PAMs), including tapered labyrinthine structures and porous materials, have been proposed based on the generalized Snell’s law. Circular-interface types of acoustic metamaterials (CAMs), however, have not been studied extensively. These can find applications on aero-engine intakes and aircraft fuselages. In this study, CAMs are studied analytically and numerically. An analytical expression of the sound refraction through the CAMs is derived based on the principle of stationary phase. In the numerical study, several fibrous metal foams with various parameters that generate a periodical linear phase gradient on the transmitted interface, are applied on circular interfaces. Results indicate that two parameters, which are the angular distance over a period of CAMs and the ratio between the radius of a circular interface and the incident wavelength, are determining factors in the excitation of high-order wave modes in the scattered sound pressure field. The high-order wave modes will disappear in the refracted sound pressure field when these two parameters are at critical values. The analytical expression is derived based on the assumption that the distance between two incident sound wave paths is infinitesimal. The assumption leads to the differences between the critical values given by analytical predictions and numerical simulations, which can be significantly reduced with the increase of ratios between the radius of a circular interface and the incident wavelength. This study offers a technical method in extending the industrial applicability of acoustic metamaterials from a planar interface to a circular interface.

Keywords: Acoustic metamaterials, fibrous materials, circular interface.

1. Introduction

PAMs, which generate a periodical linear phase gradient by assembling different materials or structures, have been widely studied. Several PAMs were designed using the generalized Snell’s law, such as the space-coiling structures [1], the tapered labyrinthine structures [2], the hexagon structures [3], the porous materials [4], and the noble gases [5]. These PAMs are proposed to achieve acoustic lens and sound absorbers. Circular-interface types of structures are widely used in aero-engine intakes, air conditioner ducts, aircraft fuselages, and so on. However, there only exist several investigations, such as the paraxial approximation of spherical metasurface [6] and curved acoustic skin cloak [7], in studying CAMs. In this study, the preliminary theory derivation, the simulation setup, and the material preparation are studied to give a technical method in engineering applications.
In this study, the analytical expression of the sound refraction through a circular interface is derived in Section 2. The designed CAM and the simulation methods are presented in Section 3. The numerical results including the predictions of critical values, the influence of determining parameters, and the anomalous phenomena are presented in Section 4. Finally, a conclusion is given in Section 5.

2. An analytical model of CAMs

A ray of a sound wave impinges on a CAM interface, which generates a periodical linear phase gradient $\nabla \Phi$ along the transmitted interface. Two different media with the refraction indexes $n_i$ and $n_t$ are separated by the circular interface as shown in Fig. 1.

![Figure 1: Schematic of a model of CAM.](image)

As the principle of stationary phase stated, the derivative of the phase accumulated along the actual sound wave path from point $P_i$ to point $P_t$, is zero for infinitesimal variations of the path [6]. An analytical model is derived based on an assumption that the distance between two incident sound wave paths is infinitesimal. So the analytical model is accurate only when the ratio between the radius of a circular interface and the incident wavelength is infinite. The phase difference between the sound wave paths $P_iA_1A_2P_t$ and $P_iB_1B_2P_t$ is zero when the distance between the point $A_1$ and the point $B_1$ is infinitesimal as shown in Eq. (1).

$$\int_{P_i}^{A_1} k_i d\vec{r} + \int_{A_2}^{P_t} k_{m} d\vec{r} + \int_{P_t}^{B_1} k_{t} d\vec{r} - \int_{A_2}^{B_1} k_{m} d\vec{r} - \int_{P_t}^{B_2} k_{t} d\vec{r} = 0, \quad (1)$$

where $k_i$, $k_m$, and $k_t$ are the wavenumber vectors in the incident medium, the metasurface medium and the refracted medium respectively, and $\vec{r}$ is the position vector in space.

The difference of the transmitted phase shift accumulated through the metasurface between the point $A_2$ and the point $B_2$ is $\nabla \Phi \cdot d\vec{s}$. $\nabla \Phi$ is the transmitted phase shift gradient generated by various properties of CAM units. $\vec{s}$ is the projection of the space position vector $\vec{r}$ along the metasurface and
its absolute value is \(|\vec{d}s| = A_2 B_2\). Then the following equation can be derived:

\[
\int_{P_1}^{A_1} \vec{k}_i d\vec{r} - \int_{P_1}^{B_1} \vec{k}_i d\vec{r} + \int_{P_2}^{A_2} \vec{k}_t d\vec{r} + \int_{P_2}^{B_2} \vec{k}_t d\vec{r} + \nabla \Phi \cdot d\vec{s} = \int_{B_1}^{A_2} \vec{k}_i d\vec{r} + \int_{B_2}^{A_2} \vec{k}_t d\vec{r} + \nabla \Phi \cdot d\vec{s} = 0. \tag{2}
\]

As the distance along the circular interface between \(A_1, A_2\) and \(B_1, B_2\) is infinitesimal, it can be assumed that \(A_1 B_1\) is equal to \(A_2 B_2\), and the distance can be expressed as \(AB\). Eq. (2) can be reduced to Eq. (3).

\[
\int_{B}^{A} (\vec{k}_i - \vec{k}_t) \cdot d\vec{r} + \nabla \Phi \cdot d\vec{s} = 0. \tag{3}
\]

The position vector in space \(\vec{r}\) is equivalent to its projection vector on the circular interface \(\vec{s}\). Then Eq. (3) can be written as:

\[
\int_{B}^{A} (\vec{k}_i - \vec{k}_t) \cdot d\vec{s} + \nabla \Phi \cdot d\vec{s} = (\vec{k}_i - \vec{k}_t + \nabla \Phi) \cdot d\vec{s} = 0. \tag{4}
\]

Equation (4) is valid for all projection vectors \(\vec{s}\) on the circular interface. So the vector \((\vec{k}_i - \vec{k}_t + \nabla \Phi)\) is perpendicular to the tangent of the interface at the incident point on the interface. The above expression can be reduced to:

\[
\vec{n} \times (\vec{k}_i - \vec{k}_t + \nabla \Phi) = \vec{n} \times \nabla \Phi. \tag{5}
\]

where \(\vec{n}\) is the normal vector. In a three-dimensional cylindrical coordinate \((r, \vartheta, z)\), the normal unit vector \(\vec{n}\) on the circular interface is \(\vec{n} = (1, 0, 0)\). The phase shift gradient is \(\nabla \Phi = (\partial \Phi / \partial r, \partial \Phi / (R \partial \vartheta), 0)\) and the wavenumber vector is \((\vec{k}_i - \vec{k}_t) = (k_{ir} - k_{it}, k_{i\vartheta} - k_{t\vartheta}, 0)\). The analytical expression of the sound refraction through the CAMs is derived as below:

\[
k_{t\vartheta} - k_{i\vartheta} = \frac{d \Phi}{k_0 R \partial \vartheta}. \tag{6}
\]

where \(k_{t\vartheta} = k_0 n_t \sin(\theta_t)\) and \(k_{i\vartheta} = k_0 n_i \sin(\theta_i)\) are the wavenumbers on the \(\vartheta\) axis in the refracted medium and in the incident medium respectively. \(n_t\) and \(n_i\) are the refraction indexes in the refracted medium and in the incident medium respectively. \(\theta_t\) and \(\theta_i\) are the angles of refracted wave and incident wave respectively. \(k_0\) is the free space wavenumber. \(R\) is the radius of the circular interface. Then Eq. (6) can be written as below:

\[
n_t \sin(\theta_t) - n_i \sin(\theta_i) = \frac{d \Phi}{k_0 R \partial \vartheta}. \tag{7}
\]

The analytical expression of the sound refraction can be developed into the diffraction expression (Eq. 8) by considering the high-order wave modes \(m\) [8].

\[
n_t \sin(\theta_t) - n_i \sin(\theta_i) = m \frac{d \Phi}{k_0 R \partial \vartheta}, \tag{8}
\]

where \(m\) is the integer diffraction peak order. The analytical expression of the sound refraction is a typical format of the diffraction expression, where the integer diffraction peak order is \(m = +1\). The analytical expression Eq. (7) is accurate when the ratio between the radius of a circular interface and the incident wavelength is infinite, which is caused by the assumption that the distance between two incident sound wave paths is infinitesimal.
3. Numerical simulation

3.1 Fibrous metal CAM design

In this study, a CAM with a radius \( R \) is proposed for study at a target frequency 2,000 Hz and its configuration is shown in Fig. 2. The metasurface is a periodic structure and one period consists of four porous units with a thickness of 2 cm. Based on the study of a planar-interface porous metasurface [4], the units are composed of fibrous metal foams, and main parameters are listed in Table 1. The adjustable geometric parameters, which are fiber diameter and porosity, are designed to form a linearly varied phase gradient on the transmitted interface, which can be expressed as \( \phi, \phi + \pi/2, \phi + \pi \) and \( \phi + 3\pi/2 \). In the simulation, the porous medium is characterized by the Johnson-Champoux-Allard model (JCA model) containing five micro-structure parameters: porosity, flow resistivity, tortuosity, viscous characteristic length, and thermal characteristic length [9, 10, 11]. These parameters of fibrous units have been obtained by a "bottom-up approach" [12][13][14] and have been validated in the previous study [4].

Table 1: Properties of the fibrous metasurface [4].

<table>
<thead>
<tr>
<th>Element</th>
<th>Fiber diameter</th>
<th>Porosity</th>
<th>Flow resistivity</th>
<th>Tortuosity</th>
<th>Viscous characteristic length</th>
<th>Thermal characteristic length</th>
</tr>
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<tbody>
<tr>
<td>Unit 1</td>
<td>8</td>
<td>0.83</td>
<td>1221565.433</td>
<td>1.086</td>
<td>1.12E-05</td>
<td>1.95E-05</td>
</tr>
<tr>
<td>Unit 2</td>
<td>8</td>
<td>0.77</td>
<td>2276785.396</td>
<td>1.120</td>
<td>7.80E-06</td>
<td>1.34E-05</td>
</tr>
<tr>
<td>Unit 3</td>
<td>28</td>
<td>0.85</td>
<td>79177.941</td>
<td>1.076</td>
<td>4.50E-05</td>
<td>7.93E-05</td>
</tr>
<tr>
<td>Unit 4</td>
<td>8</td>
<td>0.90</td>
<td>489336.421</td>
<td>1.050</td>
<td>1.97E-05</td>
<td>3.60E-05</td>
</tr>
</tbody>
</table>

3.2 Simulation setup

The governing equation of the sound pressure field is Helmholtz equation. The simulation is conducted in a commercial finite element solver COMSOL Multiphysics software. The schematic setup is shown in Fig. 2.

The fibrous metal foams are described by the JCA model in the simulation. The maximum element size is one tenth of the target wavelength to ensure accurate simulations of wave dispersion.
The circular computational domain is surrounded by perfectly matched layers to minimise
the reflections from the domain boundaries. The sound source is an ideal point source located in
the centre of the circular interface. Due to the cyclic symmetry feature of this physical problem, a sector
with periodic boundaries in the circumferential direction is used in the simulation.

4. Results and discussion

4.1 Predictions of the critical values

In the analytical study, the target frequency $f$ is 2,000 Hz. The incident angle of each point on the
circular interface is $\theta_i = 0^\circ$. The media in the incident and refracted spaces are air, and the refraction
indexes are equal to 1. The sound speed $c$ is 343 m/s. The free space wavenumber is $k_0 = 2\pi f / c$. The
angular distance over a period of CAMs is $\vartheta_p$, which is expressed as the unit angular distance in this
study. The transmitted phase shift over one unit angular distance is $2\pi$. The refracted angle $\theta_t$ can be
derived using Eq. (8).

$$\theta_t = \arcsin(\frac{c}{fR\vartheta_p}).$$

(9)

Thus, when the refracted wave angle $\theta_t$ is larger than $90^\circ$, the dominant refracted wave with
mode $m = +1$ will be converted to the surface evanescent wave. The corresponding radius $R_t$ is
the analytically critical radius. Here, the analytically critical dimensionless radius $R_t/\lambda$, which is the
ratio between the analytically critical radius of the circular interface $R_t$ and the incident wavelength
$\lambda$, is used for study. Then the expression between the analytically critical dimensionless radius $R_t/\lambda$
and the unit angular distance $\vartheta_p$ can be obtained:

$$\frac{R_t}{\lambda} \vartheta_p = 1.$$ 

(10)

In the simulation, once the fluctuation of the sound pressure level (SPL) $A$ (Eq. (11)) is smaller
than 0.3%, it can be seen that there exists only a directly transmitted wave in the scattered sound
pressure field [15]. The corresponding radius of the circular interface $R_s$ is defined as the simulated
critical radius. The simulated critical radius $R_s$ is obtained by calculating the fluctuation $A$ at the radius $R = R_s + 2\lambda$.

$$A = \frac{\text{SPL}_{\text{max}} - \text{SPL}_{\text{min}}}{\text{SPL}_{\text{ave}}}, \quad (11)$$

where $\text{SPL}_{\text{max}}$, $\text{SPL}_{\text{min}}$ and $\text{SPL}_{\text{ave}}$ are the maximum, minimum and average SPLs respectively.

The analytically and simulated critical dimensionless radii for various unit angular distances are given in Fig. 3. Figure 4 indicates that the compensation factor $k = R_t/R_s$, which is the ratio between the analytically critical radius and the simulated critical radius, is significantly reduced with the increase of $R/\lambda$. The trend agrees well with the accuracy statement in the analytical derivation.

In engineering applications, the radius of the circular-interface structure $R_d$ and the target frequency $\lambda_d$ are usually given. The corresponding analytical unit angular distance $\vartheta_d^t$ can be calculated by using Eq. (10) with the known dimensionless radius $R_d/\lambda_d$. In addition, the compensation factor $k_d$ for the dimensionless radius $R_d/\lambda_d$ can be obtained in Fig. 4. Then the simulated unit angular distance is $\vartheta_s^d = \vartheta_d^t/k_d$. So the metasurface with the unit angular distance smaller than $\vartheta_s^d$ can be applied on this circular-interface structure to reduce the SPLs in the scattered sound pressure field. To validate the disappearance of high-order wave modes at the critical values, a series of numerical simulations are implemented in the following sections.

4.2 Influence of the critical unit angular distance on refracted phenomena

Three unit angular distances are simulated at $R/\lambda = 85$ to investigate the influence on the refracted sound pressure field. The scattered sound pressure fields are given in Fig. 5.

![Figure 5: Scattered sound pressure fields at 2,000 Hz and $R/\lambda = 85$.](image)

(a) Case 1: $\vartheta_p = 0.0118$ rad.  (b) Case 2: $\vartheta_p = 0.0107$ rad.  (c) Case 3: $\vartheta_p = 0.0096$ rad.

The analytical prediction and simulated results of the critical unit angular distances are $\vartheta_t = 0.0118$ rad and $\vartheta_s = 0.0107$ rad respectively, which can be obtained from the blue point 1 in Fig. 3. There exists a difference between the analytical prediction and the simulated results. Once the unit angular distance $\vartheta_p$ is larger than the simulated critical unit angular distance $\vartheta_s$, the interference of several wave modes appear in the scattered sound pressure field, which results in the non-uniform sound pressure patterns as shown in Fig. 5a. With the decrease of the unit angular distance, the anomalous sound wave with high-order modes is converted to the surface evanescent waves at $\vartheta_s = 0.0107$ rad as shown in Fig. 5b. For the unit angular distance smaller than $\vartheta_s = 0.0107$ rad, the scattered sound pressure field keeps uniform as shown in Fig. 5c.

4.3 Influence of the critical dimensionless radius on refracted phenomena

Three dimensionless radii for two unit angular distances $\vartheta_{p1} = \pi/120$ and $\vartheta_{p2} = \pi/20$ are simulated respectively to investigate the influence of the dimensionless radius $R/\lambda$ on the scattered sound pressure fields as given in Fig. 6 and Fig. 7.
The analytical prediction and simulated results of the critical dimensionless radii are $R_{t1}/\lambda = 38.20$ and $R_{s1}/\lambda = 34.66$ respectively for $\vartheta_{p1} = \pi/120$ (green point 1 in Fig. 3). The compensation factor is about $R_{t1}/R_{s1} = 1.10$ (green point 1 in Fig. 4). When the dimensionless radius $R/\lambda$ is larger than the simulated critical dimensionless radius $R_{s1}/\lambda$, the scattered sound pressure field will be non-uniform sound pressure patterns in Fig. 6a. Once the dimensionless radius is reduced to the simulated critical value $R_{s1}/\lambda = 34.66$, the anomalous sound wave is converted to the surface evanescent wave as shown in Fig. 6b. There only exists the directly transmitted wave with mode $m = 0$ for smaller dimensionless radii as shown in Fig. 6c.

For the unit angular distance $\vartheta_{p2} = \pi/20$, the analytical prediction and simulated results of the critical dimensionless radii are $R_{t2}/\lambda = 6.37$ and $R_{s2}/\lambda = 4.60$ respectively (green point 2 in Fig. 3). The compensation factor will be increased to $R_{t2}/R_{s2} = 1.38$ (green point 2 in Fig. 4). Similar refracted phenomena such as the existence of the critical dimensionless radius and the anomalous sound pressure field can be observed in Fig. 7.

5. Conclusion

CAMs are studied analytically and numerically in this study. An analytical expression of the sound refraction through the CAMs is derived based on the principle of stationary phase. In the simulation, fibrous metal foams that generate a periodical linear phase gradient are used to study the refracted sound pressure field through circular interfaces by finite element methods.

Two parameters, the unit angular distance $\vartheta_p$ and the dimensionless radius $R/\lambda$, are analysed analytically and numerically. The results show that the appearance of high-order wave modes in the scattered sound pressure field can be tuned by adjusting these two parameters. There are critical values for these two parameters, where the sound waves with high-order modes are converted to the surface...
evanescent waves. In addition, the ratios between the critical radii given by analytical predictions and numerical simulations are defined as the compensation factors. A method is proposed to design the metasurface for given radii and frequencies by considering compensation factors. The designed metasurface can be used to reduce the SPLs in the refracted sound pressure field. This study provides an engineering method to apply CAMs in industrial applications.

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