In this work, a new Random-Eddy-Superposition (RES) technique is proposed to reproduce the major statistics of the synthetic turbulent field. The non-Gaussian filters are directly employed instead of Gaussian superposition, which are modified to generate a two-dimensional turbulent field maintaining the key characteristics of three-dimensional velocity spectra with zero spanwise wavenumber. It can overcome the limitations of the previous digital filter methods, getting rid of the inconvenience induced by complex hypergeometric functions and Mach number-dependent eddy parameters. Moreover, the RES technique offers improvement in accuracy and cost, with the results showing good agreement with theoretical solutions and experimental results.

Keywords: synthetic turbulence, leading edge noise, modified filters, random-eddy-superposition

1. Introduction

Leading edge noise can be regarded as a significant source of engine fan broadband noise, which arises from the interaction between oncoming turbulence and fan blades or Outlet Guide Vanes (OGVs). It is also known as turbulence-airfoil interaction noise, which has been investigated using analytical, computational and experimental techniques. Due to the limitations of the analytical models based on flat plate theory [1] or semi-analytical corrections [2] for considering the real geometry effect, and the limited acoustic observation angle upstream of the airfoil in experimental measurements [3], numerical studies play an important role in leading edge noise predictions. The high computational cost makes Direct Numerical Simulation (DNS) and Large Eddy Simulations (LES) infeasible in engineering field which requires high frequency resolution for the noise predictions. Thus, synthetic turbulence methods have been used to reproduce the turbulence statistics, which will then be combined with linearized Euler equations to simulate the noise generation and propagation.

The existing methods of generating synthetic turbulence can be classified as Fourier mode method, digital filter method and synthetic eddy method. Through the summation of Fourier modes, Kraichnan [4] reproduced the Gaussian energy spectrum. However, Gaussian energy spectrum is unable to represent the energy incorporated in the inertial subrange. Therefore, non-Gaussian energy spectra should be realized to describe the spectrum features in the inertial subrange. For example, on the basis of Kraichnan’s Fourier modes method [4], Karweit et al. [5] realized the von Kármán energy spectrum obeying the famous -5/3 power law in the inertial subrange. In the area of turbulence-airfoil interaction noise, Gill et al. [6] used a finite summation of discrete Fourier modes to reproduce a one-, two-, and three-component turbulent field, in order to investigate the leading edge noise for real airfoils. Apart from Fourier modes summation, synthetic eddy method was presented to synthesize...
turbulent inflow as well, which was initially proposed in Jarrin et al.’s work [7], where the turbulence can be regarded as a superposition of coherent structures defined by a shape function. By superposing the influence of randomly distributed eddies, Kim and Haeri [8] reproduced the von Kármán energy spectrum to investigate the influence of wavy leading edge on turbulence-airfoil interaction noise. However, the optimisation of 15 eddy constraint parameters made this method complicated in realizing the von Kármán energy spectrum.

Digital filter method has also been used to reproduce the specific turbulence statistics. Ewert [9] developed the Random Particle-Mesh (RPM) technique to investigate the slat noise, reproducing the local characteristics of turbulent field. Although the original RPM technique was developed to represent Gaussian energy spectrum through the use of Gaussian filters, extensions have been made to realize non-Gaussian energy spectra. Wohlbrandt et al. [10] reproduced the von Kármán energy spectrum through the superposition of 10 weighted Gaussian spectra with different length scales. Similar strategy was employed in Gea-Aguilera et al.’s work [11], where the von Kármán energy spectrum was obtained by superposing five different Gaussian eddies with Mach number-dependent eddy parameters, which is not convenient for the numerical implementation. In addition, Dieste and Gabard [12] derived the non-Gaussian filters which can be directly used to reproduce the target isotropic energy spectra. However, the two-dimensional non-Gaussian filters are hypergeometric functions whose derivatives have singularities at the center of eddy. Thus, the derivatives of the non-Gaussian filters were interpolated instead of using the exact expressions, which will introduce truncation error.

The current work presents a new RES technique of reproducing the turbulence statistics, which is based on the framework of RPM method. In the RES technique, modified Liepmann filter and von Kármán filter are directly employed instead of Gaussian superposition, getting rid of the troubles in determining different sets of eddy parameters with Mach number dependence. The modified non-Gaussian filters are developed for pseudo three-dimensional turbulence, where the key features of three-dimensional velocity spectra with zero spanwise wavenumber can be maintained by a two-dimensional Computational AeroAcoustic (CAA) simulation. The modified non-Gaussian filters show better smoothness around the barycenter of each element, compared with the two-dimensional filters with obvious singularities at the eddy center. Moreover, the modified Liepmann filter takes the form of an exponential function, which can be directly employed and easily implemented in the CAA simulation instead of the interpolation and approximation of complex hypergeometric functions. Compared with the numerical results given by existing digital filter method, the RES technique offers improvement in accuracy and computational cost, with the results showing good agreement with benchmarking analytical solutions and experimental data.

The content is organised as follows. Section 2 introduces the derivations of the modified filters. Then, the synthetic turbulent field induced by the modified filters is presented in Section 3. In Section 4 and Section 5, the RES technique is validated through the comparison with analytical solutions and experimental results. The improvement in computational cost is shown in Section 6.

2. Modified filters of non-Gaussian energy spectra

The experimental results could not be the direct object of comparison for the computational results from the synthetic turbulent field which is controlled by the filters of purely two-dimensional turbulence. Corrections factors should be introduced to link up the two-dimensional velocity spectra and the three-dimensional velocity spectra. Therefore, digital filters are modified to generate pseudo three-dimensional turbulence retaining the characteristics of three-dimensional velocity spectra, where the spanwise wavenumber is assumed to be zero following Amiet’s work [1].

For two-dimensional turbulence, the relation between the velocity spectrum and the energy spectrum can be expressed as

\[
\phi^{2D}_{11}(k_x, k_y) = \frac{E^{2D}}{\pi k} \left( 1 - \frac{k_x^2}{k^2} \right). \tag{1}
\]
In three dimensions, under the hypothesis of \( k_z = 0 \), the relationship between them becomes

\[
\phi_{11}^{3D}(k_x, k_y, k_z = 0) = \frac{E_{11}^{3D}}{4\pi k^2} \left( 1 - \frac{k_z^2}{k^2} \right).
\] (2)

According to Eqs. (1) and (2), if \( E_2^{2D} \) is in accordance with \( E_3^{3D}/4k \), the velocity spectrum of two-dimensional turbulence, \( \phi_{11}^{2D}(k_x, k_y) \) will be in conformity to \( \phi_{11}^{3D}(k_x, k_y, k_z = 0) \), which is also applicable to the transverse unsteady velocity.

The relation between the energy spectrum function and the filter can be written as [12]

\[
E_{2D}(k) = \frac{k^3}{4\pi} \hat{G}(k)^2, \tag{3}
\]

where \( \hat{G}(k) \) denotes the Fourier transform of the filter in wavenumber space. Hence, if \( E_2^{2D} \) is in line with \( E_3^{3D}/4k \), \( \hat{G}_M(k) \) will take the form

\[
\hat{G}_M(k) = \sqrt{\frac{\pi E_3^{3D}(k)}{k^2}}, \tag{4}
\]

where \( \hat{G}_M(k) \) denotes the Fourier transform of the modified filter for pseudo three-dimensional turbulence. Using the properties of Fourier-Bessel transform yields the modified filter in physical space,

\[
G_M(r) = \frac{1}{2\pi} \int_0^\infty k \hat{G}_M(k) J_0(kr) dk. \tag{5}
\]

The modified filter for pseudo three-dimensional turbulence is obtained when inserting Eq. (4) into Eq. (5) yielding

\[
G_M(r) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \sqrt{\frac{E_3^{3D}(k)}{k}} J_0(kr) dk. \tag{6}
\]

The Liepmann energy spectrum in wavenumber space is defined as

\[
E_L^{3D}(k) = \frac{8}{\pi} u_{rms}^2 \lambda^5 \frac{k^4}{(1 + 2\lambda^2 k^2)^{17/6}}, \tag{7}
\]

where \( u_{rms} \) stands for the root-mean-square of the velocity fluctuation and \( \lambda \) denotes the integral length scale. The modified filter of the Liepmann energy spectrum can be obtained by inserting Eq. (7) into Eq. (6), which yields in physical space,

\[
G_M^L(r) = \frac{\sqrt{2}}{\pi} u_{rms} \sqrt{\lambda} e^{-r/\lambda}. \tag{8}
\]

The three-dimensional von Kármán energy spectrum is given by

\[
E_v^{3D}(k) = \frac{55}{9\pi} u_{rms}^2 \lambda^4 \frac{k^4}{(1 + \zeta^2 k^2)^{17/6}}, \tag{9}
\]

where \( \zeta = \frac{\Gamma(1/3)}{\sqrt{\pi} \Gamma(5/6)} \lambda. \) Using Eq. (6), the modified von Kármán filter in physical space reads,

\[
G_v^M(r) = 0.29 u_{rms} \lambda^{0.083} r^{0.42} K_{-0.42} \left( \frac{\sqrt{\pi} \Gamma(5/6)}{\lambda \Gamma(1/3)} \right), \tag{10}
\]

where \( K \) denotes the modified Bessel function of the second kind.

Figure 1 shows the characteristics of the modified filters for Liepmann energy spectrum and von Kármán energy spectrum, where the modified filters show good smoothness around the eddy center compared with the original two-dimensional filters without modification [12]. This implies that the singularities at the origin are removed to some extent by the implementation of pseudo three-dimensional turbulence, which makes it possible to reproduce the major statistics of the turbulence by using the filters directly.
3. Stochastic field induced by the modified filters

Following the Lagrangian discretisation [12], the two-dimensional synthetic turbulent field can be regarded as the superposition of several fluid elements or eddies, i.e., \( V_e = \sum_{s=1}^{S} V_{e,s} \). The fluctuating velocity field can be defined as

\[
\begin{align*}
    u'(x, t) &= \sum_{s=1}^{S} \frac{\partial G^M(|x - x_c|)}{\partial y} \int_{V_{e,s}} U(x_e, t) \, dx_e, \\
    v'(x, t) &= -\sum_{s=1}^{S} \frac{\partial G^M(|x - x_c|)}{\partial x} \int_{V_{e,s}} U(x_e, t) \, dx_e,
\end{align*}
\]

where \( G^M(|x - x_c|) \) denotes the modified filter, \( x \) represents the position vector at a certain point in the turbulent flow field and \( x_c \) is the eddy center of each fluid element \( V_{e,s} \). The last item in the above expression denotes the weighted average of the white noise term over each fluid element \( V_{e,s} \), which can be named as \( U_s \) for the sake of brevity. Taking the properties of the white noise term into consideration, \( U_s \) can be replaced by \( \Omega \Delta \) following the distribution whose mean value is 0 and standard deviation is \( \Delta \) [11], where \( \Delta \) denotes the distance between the vortex centers. Therefore, the fluctuating velocity field induced by each vortex can be expressed as,

\[
\begin{align*}
    u'(x) &= \frac{\partial G^M(|x - x_c|)}{\partial y} \Omega \Delta, \\
    v'(x) &= -\frac{\partial G^M(|x - x_c|)}{\partial x} \Omega \Delta,
\end{align*}
\]

where \( \Omega \) takes the values +1 or −1 stochastically, which can be considered as a parameter related to the direction of rotation for each vortex.

In the realisation of the Liepmann energy spectrum, the synthetic turbulent field introduced by each vortex can be obtained when Eq. (8) is introduced, which yields

\[
\begin{align*}
    u'(x) &= -\frac{\sqrt{2}}{\pi \sqrt{\lambda}} \frac{(y - y_c)}{r} e^{-r/\lambda} u_{rms} \Omega \Delta, \\
    v'(x) &= \frac{\sqrt{2}}{\pi \sqrt{\lambda}} \frac{(x - x_c)}{r} e^{-r/\lambda} u_{rms} \Omega \Delta.
\end{align*}
\]
4. Verification case: NACA 0001 airfoil

In this section, the RES technique with modified filters is verified by comparing with the analytical solution based on Amiet’s theory [1]. The numerical simulation is implemented on a NACA 0001 airfoil whose chord length is $c = 0.15 \text{ m}$ and span length is $2d = 0.45 \text{ m}$. During the simulation, the turbulent velocity field induced by the modified filters can be regarded as the pseudo three-dimensional turbulence, which will reproduce the major statistics of the Liepmann energy spectrum whose turbulent integral length scale $\lambda$ is 0.008 m and turbulence intensity is 0.017. There are two primary parameters in characterising the synthetic turbulence generated by these modified Liepmann eddies. One is the distance between the eddy centers, which is set to $\Delta \leq \lambda/8$, the other is the influence region of the eddies, which is determined by $r_e \geq 5\lambda$ according to the decay characteristics of the modified filter. The Liepmann eddies are convected by the background mean flow whose Mach number is 0.6.

The validity of the obtained turbulent velocity field can be evaluated by the one-dimensional spectra of an observation point placed upstream of the leading edge. The sampling rate of the fluctuating velocity field is 250 time steps, which is set to $1.18 \times 10^{-8} \text{ s}$ for each time step. As shown in Figure 2, the one-dimensional energy spectra calculated by the RES technique shows good agreement with the analytical results, which implies that the RES technique produces satisfactory major statistics of the turbulent flow field.

![Figure 2: One-dimensional energy spectra of the observation point.](image)

The far-field noise can be predicted using a three-dimensional Ffowcs-Williams and Hawkings (FW-H) solver, which collects the unsteady information on the airfoil surface in a two-dimensional computational domain. Then the fluctuating components of the three-dimensional airfoil can be obtained through the reproduction along the spanwise direction. After solving the three-dimensional FW-H equation, a $\pi/d$ factor should be introduced to modify the far-field acoustic power spectral density when the spanwise wavenumber is assumed to be 0 in turbulence modelling [13]. As can be seen from Figure 3, Sound Pressure Level (SPL) spectra at various observation points are in good agreement with Amiet’s theory [1], which means that the RES technique shows excellent performance in predicting the far-field noise for three-dimensional airfoils based on the implementation of pseudo three-dimensional synthetic turbulence.

5. Validation case: NACA 0012 airfoil

In order to further validate the accuracy of modified filters developed for pseudo three-dimensional turbulence, the numerical results obtained from the modified filter of von Kármán energy spectrum are
then compared with the experimental results collected by Paterson and Amiet [14] in an open-circuit wind tunnel test, where a NACA 0012 airfoil with 0.23 m chord and 0.53 m span was subjected to the incoming flows whose turbulent integral length scales were 0.03 m and velocities were 60 m/s, 90 m/s and 120 m/s, respectively. Figure 4 shows the far-field noise predictions at zero incidence for an observer at $\theta = 90^\circ$, where the numerical results given by the RES technique show good agreement.
with the experimental data at \( U_x = 60 \text{ m/s} \). Although the disagreement seems to be larger when the speed of inflow increases, the maximum deviation is relatively small compared with the numerical results from the advanced digital filter method [11] especially for the low-frequency range. For the 120 m/s case, the experimental data and numerical results presented in this paper are within 2 dB, which is only half of the largest disagreement in Gea-Aguilera et al.’s implementation [11] under the same circumstance. For the high-frequency range, the disagreement may be owing to the 2-4 dB corrections in view of the effect of tunnel background noise in Paterson and Amiet’s experiment [14], which could lead to great uncertainty to some extent.

6. Computational cost

In this section, the computational expense of the RES technique is compared with that of the advanced digital filter method [11], which is based on the same computational grid and processor resources. As shown in Table 1, the simulation runs faster with the RES technique per time step, which is approximately 2.5 hours faster than the advanced digital filter method totally.

This computing speed enhancement can be explained from two aspects. On the one hand, in the advanced digital filter method, the fluctuating velocity field was obtained through the superposition of five Gaussian filters with Mach number-dependent parameters. However, only one non-Gaussian filter is directly used at one control point in the current technique, which can accelerate the initialization of unsteady velocity field. On the other hand, the influence region of the filter is chosen as \( r_e = 5\lambda = 0.04 \) in the current technique, which is smaller than the influential radius \( r_e = 5 \max \{\Lambda_i\} / 2 = 0.09045 \) in the advanced digital filter method [11], where \( \max \{\Lambda_i\} \) denotes the maximal length scale among the five eddies. This indicates that the affected area and the inlet region used for introducing the eddies are thinner in the current method, which can save the computational cost.

<table>
<thead>
<tr>
<th>Total time elapsed (s)</th>
<th>Time per time step (s)</th>
<th>Total time steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>RES technique</td>
<td>31515</td>
<td>0.06303</td>
</tr>
<tr>
<td>Advanced digital filter</td>
<td>40625</td>
<td>0.08125</td>
</tr>
</tbody>
</table>

Table 1: Comparison of computational cost

7. Conclusions

The RES technique is used in this work to realize the major statistics of the synthetic turbulent field. Instead of reproducing the non-Gaussian energy spectra through Gaussian superposition, Liepmann filter and von Kármán filter are directly employed in this work, which are modified to reproduce a two-dimensional turbulent field which maintains the main features of three-dimensional velocity spectra, where the spanwise wavenumber is assumed to be zero following Amiet’s theory. The modified non-Gaussian filters show better smoothness around the eddy center than the original filters with apparent singularities at the vortex core, which means that the modified filters can be used in the numerical implementation directly instead of subsequent interpolation and approximation. The RES technique is verified through the comparison with Amiet’s analytical solution for a flat plate. By solving the three-dimensional FW-H formulation, the far-field SPL spectra is obtained, which shows good agreement with the theoretical solution within 1.5 dB accuracy. This technique also performs well in predicting the far-field noise from a NACA 0012 airfoil, which is based on the modified filters for pseudo three-dimensional turbulence. Considering the cases for different Mach numbers, we draw the conclusion that the numerical results and experimental data are within 2 dB, which is potentially affected by the 2-4 dB background noise corrections in the experiments. In addition, the calculation
expense is reduced compared with that of the advanced digital filter method, which is attributed to the
direct implementation of non-Gaussian filters and the shrunken influence region.

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