In this paper, the energy flow model for functionally graded (FG) plates is developed to predict the high frequency dynamic behavior. The material properties of the FG plates are assumed to vary continuously in the thickness direction according to the power-law form. The dispersion relation for the bending wave in FG plates is derived and the wavenumber is obtained. The energy flow model for reverberant wave field is firstly derived based on the assumption that the wave field can be expressed as the superposition of the plane waves. Following the formation mechanism of the wave field in the two-dimensional structures, the wave field in FG plates is then separated to the direct wave field coming from the driving points and the reverberant field formed by the wave scattering on plate boundaries. The wave scattering and the perfectly reflected boundary conditions are used to link the direct wave field and the reverberant wave field and a ray tracing technology is used to derived the boundary condition for the energy flow model for reverberant wave field. The energetic response for the FG plates is obtained by accounting for both the direct wave field and the reverberant wave field. To validate the proposed energy flow model, the energy levels from the energy flow model are compared with those from the modal solution for different cases. Both good agreements and accuracy between the proposed model and the modal solution can be observed for all the cases, which are more apparent than those between the classical energy flow model and the modal solution. The results show that the hybrid energy flow model is suited to the high frequency analysis of the FG plate.

Keywords: Functionally graded material, Plate vibration, High frequency vibration, Energy finite element analysis

1. Introduction

Functionally graded material (FGM) is an emerging type of composite material which was put forward by the Japan material scientist in 1980s for a space project. FGM are usually made up of two types of the constituent; the metal material and the ceramic material. Due to the high temperature resistance of the ceramic material, the FGM has the ability to resist the high temperature and thus can be used in the high temperature environment. Meanwhile, the strong mechanical performance of the metal constituent can improve the endurance of the FGM. Different from the laminated composite materials, the transition between the different material constituents is gradual and the interfaces of the different material components are eliminated. The gradual transition of the different material components as well as the material properties can effectively reduce the risk of delamination and cracking which can be often observed in the laminated composite materials, especially in the structures under the large thermal gradient. In addition, the volume fractions of the material constituents can be designed and optimized in accordance to the specific requirements. Owing to the advantages of the FGM in many aspects, the FGM recently are not only used for the thermal barrier system but also have
been increasingly used for structural components in many other fields such as turbine blade, weapon armor and the aerospace industry, and so on. It can be expected that the FGM will be utilized in many more fields with the development of the FGM.

The plates made of the FGMs are applied as the structural components in many fields. The dynamic response of the plate structures is of significance for the vibroacoustic behavior of the products because the vibrating plate would compress the liquid such as the air and the water and radiate the structure-borne noise, which is a main source for the structure-borne noise for many products. With the increased custom expectations and the restrictive legal regulations on the vibroacoustic behavior of the product, the prediction of the vibration of the plate structures become an indispensable part in the produce design process.

There exist several prediction methods for the plate vibration, among which the Finite Element Method (FEM) may be the most popular one due to its ability to handle the region with the complexities [1]. The FEM can be used to analyze the dynamic response at low frequency. However, the FEM is not an ideal tool for mid-high frequency analysis because FEM is time-consuming and not reliable at mid-high frequencies. At high frequency, the statistical energy analysis (SEA) is a widely used tool with many industrial applications [2]. However, since the basic element in SEA is the subsystem, the spatial variation of the response can-not be obtained from SEA. Moreover, as the SEA is a lump method, the local effects such as the local damping and local power cannot be modeled readily.

As an alternative to SEA, the energy flow analysis (EFA) is an emerging method for high frequency analysis. The major advantage of EFA over the SEA is that the EFA is capable of predicting the spatial variation of the response. Due to the advantages of EFA in many aspects, many researches are dedicated in this method. Up to now, the EFA can be used to analyze many kinds of structures such as rods, beams, membranes, plates, shells and coupled structures [3–6]. However, most of the energy flow model are developed for the homogeneous structures while few researches concern the FG structures. The energy flow model for FG beams is developed by Liu and Niu [7] whereas that for plate structures have not been investigated yet. The classical energy flow model for two-dimensional structures, as demonstrated by Langley [8], Smith [9] and Kong [10], is established based on the analogy between the mechanical energy flow and the thermal energy flow. This analogy is valid only if the reverberant wave assumption of the EFA is satisfied. However, when such requirement is not met, the prediction error of the EFA is significant. To fix this issue and make the EFA applicable for more situations, the formation mechanism of the wave field in the plate should be followed and the contribution of the direct wave should also be taken into account. Thus, this paper aims to develop a hybrid energy flow model for FG plates. Different from the traditional energy flow model, the direct wave will be considered in the study by using a ray tracing technology.

2. Energy flow model for FG plates

2.1 Description for the FG plates

Consider a FG plate with the length \(a\), width \(b\) and thickness \(h\), as shown in Fig. 1. The coordinate system is established on the mid surface of the plate in which the \(z\) axis is normal to the plane of the plate. \(u\), \(v\) and \(w\) denote the deformation in the \(x\), \(y\) and \(z\) direction, respectively. The FGM is made up
of two different material constituents. The material properties of the FG plates, including the Young’s modulus $E$, Poisson’s ratio $\mu$, mass density $\rho$, are assumed to vary smoothly and continuously in the thickness direction estimated by the Voigt’s rule of the mixture, namely, the material properties are the linear combination of the constituent properties weighted by the volume fraction. According to the Voigt’s rule of the mixture, the effective material property of the composite material can be expressed as

$$ P = P_t V_t + P_b V_b $$

(1)

where $P_t$ and $P_b$ are the properties of the material on the top and bottom surface of the FG plate, $V_t$ and $V_b$ are the volume fraction of the material constituents on the top surface and the bottom surface. There are several different math models to describe the material volume of the FG structures. In this work, the material volume fraction is determined by the power-law form. For the power-law form, the volume fraction of the material on the top surface can be expressed as

$$ V_t = \left( \frac{z}{h} + \frac{1}{2} \right)^n $$

(2)

where $h$ is the thickness of the plate, $n$ is the power-law exponent which is a positive number indicating the different transition between the two different material components. The variations of the volume fraction for different $n$ are shown in Fig. 2.

![Figure 2: Volume fraction $V_t$ for different power-law exponents](image)

2.2 Derivation for governing for EFA

Based on the Kirchhoff plate theory and using the equilibrium approach, the governing equation for the FG plate bending motion can be derived as

$$ D \frac{\partial^4 w}{\partial x^4} + D \frac{\partial^4 w}{\partial y^4} + 2 \left( D_{xy} + D_k \right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\rho h}{\partial t^2} = f_z (x, y, t) $$

(3)

where $f_z$ is the transverse force applied on the plate with the distributed form. If the applied force is in the concentrated form, the delta function can be used to accommodate the concentrate force, $D$, $D_{xy}$ and $D_k$ are the bending stiffness coefficients, $\rho h$ is the inertia coefficients. These coefficients can be expressed as

$$ D = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) z^2 \, dz, \quad D_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z) \mu(z) z^2}{1 - \mu(z)^2} \, dz, \quad D_k = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z) z^2}{1 + \mu(z)} \, dz, \quad \rho h = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z) \, dz $$

(4)

The FGM is regarded as inhomogeneous and isotropic. It can be observed from Eq. (4) that $D$, $D_{xy}$ and $D_k$ are related by

$$ D = D_{xy} + D_k $$

(5)

Substitution of Eq. (5) into Eq. (3), the governing equation for the bending motion of the FG plate can be rewritten as

$$ D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = f_z (x, y, t) $$

(6)
It can be observed that Eq. (6) is in the same form as that for the homogeneous plate based on the Kirchhoff plate theory. The general solution for Eq. (6) is
\[ w(x, y, t) = A_n e^{i k x} e^{i k y} e^{i \omega t} \]
where \( k_x \) and \( k_y \) are the wavenumber components along the \( x \) and \( y \) direction, respectively, indicating the oscillation frequency of the wave with respect to the space domain, \( \omega \) is the circular frequency indicating the wave oscillation frequency with respect to the time domain. Substitution of Eq. (7) into Eq. (6), the dispersion relation for the bending wave of the FG plate can be obtained as
\[ D k^2 x + k^2 y - \rho \omega^2 = 0 \]
The dispersion relation for the bending wave can be expressed as
\[ D k^4 - \rho \omega^2 = 0 \]

The general solution for Eq. (6) can be expressed as
\[ w(x, y, t) = (A_1 e^{i k x} + A_2 e^{-i k x}) (B_1 e^{i k y} + B_2 e^{-i k y}) e^{-i \omega t} \]

It should be noted that the solution for Eq. (6) in Eq. (10) is not complete; the solution in Eq. (10) only contains the far field solution whereas the near field solution, which represents the evanescent wave, is neglected in Eq. (14). Such operations have been adopted by many other researchers concerning the vibrational power flow because the evanescent wave transmits no power. Also, the evanescent waves decay exponentially with respect to the space so that they can be neglected half wavelength away from the boundaries.

When the damping effect is considered, the effect of the damping on the wave decay with respect to the propagation in the space domain can be accommodated by using the hysteretic damping model. Cremer and Heckle show that the hysteretic damping can be incorporated into the governing equation by using the complex Young’s modulus. Since the damping is also assumed to vary in the thickness, the complex bending stiffness can be obtained by substituting the complex Young’s modulus \( E(z)(1 + j \eta(z)) \) into Eq. (4), such that
\[ D_c = \int_{-h/2}^{h/2} \frac{E(z)(1 + j \eta(z)) z^2}{1 - \mu(z)^2} \, dz = D \left( 1 + j \eta_{eff} \right) \]

where \( \eta_{eff} \) is the effective damping, which can be expressed as
\[ \eta_{eff} = \frac{\text{Im}(D_c)}{D} \]

Since the damping effect is considered by using the complex Young’s modulus, the wavenumber is also a complex number containing both the real part and the imaginary part; the real part corresponds to the wave propagation whereas the imaginary part corresponds to the wave decay with respect to the space due to the damping. The complex wavenumber can be expressed as
\[ k_c = \sqrt{\frac{\rho \omega^2}{D_c}} = k \left( 1 - j \frac{\eta_{eff}}{4} \right) \]

The energy density of the FG plates is the sum of the kinetic energy density and the potential energy density. Based on the Kirchhoff plate theory, the time averaged energy density can be expressed as
\[ e = \frac{1}{4} \left( \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial^2 w}{\partial x^2} \right)^* + \frac{\partial^2 w}{\partial y^2} \left( \frac{\partial^2 w}{\partial y^2} \right)^* + 2 D_{xy} \frac{\partial^2 w}{\partial x \partial y} \left( \frac{\partial^2 w}{\partial x \partial y} \right)^* + \frac{\rho \omega}{D} \frac{\partial w}{\partial t} \left( \frac{\partial w}{\partial t} \right)^* \right) \]

The time averaged energy intensity can be expressed as
\[ I_x = -M_s \frac{\partial^2 w}{\partial x \partial t} - M_{xy} \frac{\partial^2 w}{\partial y \partial t} + Q_x \frac{\partial w}{\partial t}, \quad I_y = -M_y \frac{\partial^2 w}{\partial y \partial t} - M_{yx} \frac{\partial^2 w}{\partial x \partial t} + Q_y \frac{\partial w}{\partial t} \]
Substituting Eq. (10) into Eq. (14) and performing the space average over a wavelength to eliminate the interface between the different waves, the time and space averaged energy density can be expressed as

$$
\langle \vec{e} \rangle = \frac{D}{4} \left( \frac{k_x^2 + k_y^2 + 2D_{xy}k_x^2k_y^2}{D} \right) \left( |A_1|^2 |B_1|^2 e^{-\eta_{eff} \frac{2}{\omega} (k_x x + k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (k_x x - k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (-k_x x + k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (-k_x x - k_y y)} \right)
$$

The time and space averaged energy intensity can be expressed as

$$
I_x = 2\omega k_x D \left( k_x^2 + k_y^2 \right) \left( |A_1|^2 |B_1|^2 e^{-\eta_{eff} \frac{2}{\omega} (k_x x + k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (k_x x - k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (-k_x x + k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (-k_x x - k_y y)} \right)
$$

$$
I_y = \omega k_y D \left( k_x^2 + k_y^2 \right) \left( |A_1|^2 |B_1|^2 e^{-\eta_{eff} \frac{2}{\omega} (k_x x + k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (k_x x - k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (-k_x x + k_y y)} + |A_1|^2 |B_2|^2 e^{-\eta_{eff} \frac{2}{\omega} (-k_x x - k_y y)} \right)
$$

From Eqs. (16) to (18), the relation between the energy density and the energy intensity can be expressed as

$$
I = -\frac{c_g^2}{\eta_\omega} \nabla e
$$

where $c_g$ is the group velocity, $c_g = 2\omega / k$. For the hysteretic damping model, the power dissipation due to the damping can be expressed as

$$
p_{diss} = \eta_{eff} \omega \langle \vec{e} \rangle
$$

Considering the power balance of the differential volume in the plate, the power entering the differential volume must equal to the power leaving and the power dissipation. Such a conservation relation can be expressed as

$$
p_{diss} + \nabla \cdot (I) = p_{in}
$$

where $\nabla \cdot$ is the divergence operator, $p_{in}$ denotes the input power due to the external force. Substituting Eqs. (19) and (20) into Eq. (21), the governing equation for the energy density of the bending wave can be derived as

$$
-\frac{c_g^2}{\eta_{eff} \omega} \nabla^2 e + \eta_\omega e = p_{in}
$$

The energy density governing equation in Eq. (22) is based on the assumption that the wave field can be expressed as the superposition of the plane waves, namely, the Eq. (10). Such an assumption is valid when the wave field is reverberant or quasi-reverberant. When the influence of the direct wave field is not negligible, the prediction error of the Eq. (22) is significant. To fix this issue, the direct wave field and the reverberant wave field can be separated and taken into account simultaneously.

As shown in Fig. 3, according to the formation mechanism of the wave field, the direct wave coming from the driving point, in essence, is first created by the driving force and the reverberant wave is formed by the wave scattering and diffraction of the direct wave on the boundary. At high frequency, the interface between the reverberant wave and the direct wave can be neglected so that the energy density can be regarded as the linear combination of these two wave fields, such that

$$
e = e_d + e_r
$$

where $e_d$ denotes the energy density of the direct wave, $e_r$ denotes the energy density of the reverberant wave. The energy density of the direct wave can be obtained by analyzing the FG plate with the infinite extent, namely, the FG plate without the reflection boundaries. The energy density governing equation for the direct wave in FG plate is

$$
\frac{d}{dr} \left( r \langle e_d \rangle \right) + \frac{\eta_{eff} \omega}{c_g} r \langle e_d \rangle = \frac{r P_{in}}{c_g}
$$
where \( r \) denotes the distance from the driving point. The solution for Eq. (24) is
\[
e_d = \frac{P_{in}}{2\pi r c_g} e^{-\frac{\eta_{eff} \omega}{r c_g}}
\] (25)
As shown in Eq. (25), the solution for Eq. (24) will be infinite and singular when \( r \to 0 \). To remove the singularity of the Eq. (25), the piecewise function describing the energy density of the direct wave can be introduced as
\[
e_d = \begin{cases} 
\left(\frac{P_{in}}{2\pi r_0 c_g}\right) e^{-\frac{\eta_{eff} \omega}{r c_g}} & r < r_0 \\
\frac{F_0^2}{128 D} & r > r_0 
\end{cases}
\] (26)
(27)
where \( r_0 \) is the radius of the singular circle [9]. The power entering the reverberant wave field can be obtained by using a ray tracing technology. As shown in Fig. 4, for the perfectly reflected boundary, the boundary condition for the reverberant wave field can be expressed as [11]
\[
\frac{c_g^2}{\eta_{eff} \omega} \nabla e_r \cdot \mathbf{n} = q_d \cdot \mathbf{n}
\] (28)
where \( q_d \) is the energy intensity of the direct wave field at the boundary, \( \mathbf{n} \) is the unity normal vector of the boundary. The energy intensity of the direct wave field impinging the boundary at the point \( P \) can be expressed as
\[
q_d (r) = c_g e_d (r)
\] (29)
where \( r \) is the distance between the source point \( S \) and the boundary point \( P \). The boundary condition of the reverberant wave field can be implemented with the Finite Element Method because the Finite Element Method is the most common used numerical method to solve the governing equation of the EFA. As shown in Eq. (28), for the perfectly reflected boundary, it is assumed that all the energy intensity is reflected to the reverberant wave field. The reflected energy intensity at the boundary contributes the residual to the element node on the boundary in the implementation of the Finite Element Method.
3. Validation of the hybrid energy flow model

In this section, the numerical simulations are performed to validate the proposed hybrid energy flow model for FG plate. The square FG plate with dimension of \( L \times b \times h = 1 \text{ m} \times 1 \text{ m} \times 0.001 \text{ m} \) and made of the stainless steel and the alumina is selected as the simulation model. The four boundaries are all assumed to be simply supported. The top surface of the FG plate is pure stainless steel whereas the bottom surface is the pure alumina and the transition between the top and bottom surface is linear by setting \( n = 1 \) in Eq. (2). The properties of these two kinds of material components can be found in Ref.[12]. In the simulations a harmonic force with the unity power is applied on the center of the FG plate and the energy levels from the modal solutions for FG plate are regarded as exact and taken as benchmarks for the comparisons. The energy level comparisons are made along the \( x \) direction across the driving point. To guarantee the convergence of the modal solution, the first 150 modes are truncated. Fig. 5 shows the comparisons between the modal solution, classical EFA and the hybrid EFA for different frequencies. The damping factor for stainless steel and the alumina are set to 0.05 and 0.06, respectively. It can be observed from Fig. 5 that for both these two cases the hybrid EFA presents the trend the modal solution. Good correlations can be found between the modal solution and the hybrid EFA. Also, it can be observed from Fig. 5 that the classical EFA underestimates the energy level near the compared with the classical EFA and overestimates the energy level near the boundary. The improvement on the accuracy of the hybrid EFA is obvious.

![Figure 5: Energy level for the FG plates for (a) \( f = 1000 \text{ Hz} \) and (b) \( f = 5000 \text{ Hz} \) when \( \eta_t = 0.05, \eta_b = 0.06 \)](image-url)

![Figure 6: Energy level for the FG plates for (a) \( \eta_t = 0.1, \eta_b = 0.1 \) and (b) \( \eta_t = 0.02, \eta_b = 0.02 \) when \( f = 3000 \text{ Hz} \)](image-url)
4. Conclusions

The hybrid energy flow model for FG plates was developed in this paper. The material properties of the FG plate are assumed to vary continuously and smoothly in the thickness direction according to the power-law forms. Based on the governing equation for the bending motion of the FG plate, the dispersion relation for the bending wave was derived and the wavenumber was obtained. The classical energy flow model was then derived for the reverberant wave field. Following the formation mechanism of the wave field on the FG plate, both the direct wave field and the reverberant wave field were taken into account simultaneously. Using the ray tracing technology, the direct wave energy intensity is regarded as the boundary condition for the reverberant wave field so that the input is injected into the reverberant wave at the boundary rather than the excitation point. The overall energy level for the FG plate was the linear superposition of the direct wave field and the reverberant wave field.

To validate the proposed energy flow model, the energy level from the modal solution were compared with the hybrid energy flow for different cases. It is found that the proposed hybrid energy flow model can present the trends of the modal solution. Both the accuracy and the correlation between the modal solution were observed, which were more apparent than those between the modal solution and the classical energy flow model. The results show that the hybrid energy flow model developed in this paper is suited to the high frequency vibration analysis for FG plates, which is the topic of concern in aerospace, marine and other mechanical system with the FG plates.

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References