PLATE VIBRATION DISPLACEMENT CURVE MEASUREMENT

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Plate dynamics are often measured using accelerometers and in some cases laser based systems. Natural frequencies, mode shapes, and deflections are then derived from these measurements. The work presented here describes a method to directly measure the deflection curve of a vibrating plate using piezoelectric films. The sensor consists of constant shape PolyVinylidene Fluoride (PVDF) films bonded to the surface of the plate. These sensors were initially developed for measuring the displacement curves of vibrating beams. Their application is extended here to vibrating plates. We show in this paper that each segment of the sensor measures the deflection slope at its location. The overall plate lateral displacement curve is calculated from the slopes using central difference formulas. In this paper, the equations of the sensor are derived along with numerical verifications. The numerical simulation is performed for plates vibrating at various frequencies including resonance and off resonance frequencies. Results for simply supported and clamped plates are presented. These results indicate that the proposed sensor can be used to effectively measure the lateral vibration displacements curves of plates with various boundary conditions.

Keywords: Plate, Sensor, PVDF, Vibration

1. Introduction

Actuators and sensors play a central role in active vibration control, acoustic emission monitoring, nondestructive testing, structure health monitoring, and many other types of applications [1]. In the last three decades or so, the design of the actuators and sensors has been focused on piezo films [2] especially on the use of PolyVinylidene Fluoride (PVDF). PVDF is a piezoelectric polymer that can be poled in thin films down to nine micron [3] \((9 \mu m)\) making them suitable for sensor development because they add little loading to the receiving structure and are easy to cut, shape, and etch[4].

Many applications of PVDF based sensors can be found in literature in active noise and vibration control [5], Material characterization [6], Medical field [7] etc. The PVDF is usually in the form a film bonded to the structure [8]. To cite but a few, recent applications of PVDF as actuator involved a vibrating membrane used for fatigue test of thin films [9]. Another application uses PVDF laminate as actuator to control the vibration of a cylindrical shell [10]. In these applications, multiple layers of film are used to increase the available actuating force. However for sensing, the ideal situation is to use as minimum number of layers as possible so that the sensor does not interfere with the structure’s dynamic properties. In general the film is very flexible compared to the structure to which it is bonded therefore the strain transferred to the structure is expected to be very small. The film is usually shaped [11] to extract the dynamic properties of interest. Researchers have measured the volume velocity of beams and plates for active noise and vibration cancellation [12] using quadratic functions to shape the sensor film while others have used a mixture of quadratic and linear functions to shape sensors that measure localized volume velocity [13].
The current paper presents the framework for the measurement of the lateral displacements of a vibrating plate using distributed sensor for various plate boundary conditions. Generally, well-established point sensors such as accelerometers are used to measure the dynamic properties of vibrating structures. However for control systems, especially since the advent and wide spread of active control and structural health monitoring, researchers have been looking for more non-conventional sensors, mainly distributed sensors. In the case of active vibration control, distributed sensors tend to provide better vibration properties of the controlled structure. Unlike point sensors, distributed sensors can give simultaneous measurement data for various locations on the structure and are less likely to miss a vibration mode. For example, a point sensor on a relatively long beam could provide a false reading of the state of the beam if its location corresponds to a vibration node [14].

One particular promising application that motivated the investigation of the proposed sensor is the monitoring of the structural supports of signs, luminaires, and traffic signals [15]. Various vibration mitigation devices have been proposed for these structural supports with no clear solution on how to assess their effectiveness. The sensor presented here could be an affordable solution for evaluating and monitoring the effectiveness of the vibration-mitigation [16] devices of structural supports of signs, luminaires, and traffic signals.

The proposed sensor can also be used in active vibration and noise control or structural members health monitoring. The sensor simultaneously measures at multiple points the slopes of the vibrating surface to yield the instantaneous real time vibration shape of the plate. Important plate mechanical entities such as strains and stresses can be readily computed from the deflection curves making the proposed sensor an invaluable asset in control and structural health monitoring. Dynamic properties such as natural frequencies and mode shapes can also be calculated from the instantaneous deflection curves.

Strain based dynamic point sensors have been around for decades. These sensors use internal beams [17] to relate the dynamic properties of the structure to the strain on the surface of the beam. The strain at any point on the cantilever beam is proportional to the deflection of the mass, therefore the displacement of the base and the motion of the corresponding point on the structure can be found. The sensor proposed here extends the cantilever beam sensor concept with the strain sensor (PVDF) directly attached to the structure. Multiple sections of the sensor is used to measure the strain at multiple locations on the plate and translate those strains into displacements on the surface of the vibrating plates. It is therefore obvious that the accuracy of the measurement will depends on the numbers of sensor sections relative to the highest target frequency [18]. The proposed displacement sensor approach presented here is made possible by the availability and cost of PVDF films, etching processes, and single chip computers.

2. Sensor design

2.1 Beam displacement sensor equation

The design and implementation of the beam lateral vibration displacement sensor can be found in literature [19]. The theory of the beam sensor is briefly reported here to provide insight in the design and development of the plate lateral vibration displacement sensor. For multiple patches of film bonded to a beam surface as shown in Fig. 1 with \( p \) segments of films, the \( i^{th} \) patch output charge can be in the form of:

\[
\phi_i = -\frac{bh'}{S_i} \int_{x_{i-1}}^{x_i} \left( h_{31} r_{x'} \frac{\partial^2 z}{\partial x^2} \right) dx . \tag{1}
\]

After integration, Eq. (1) yields:

\[
\phi_i = -\frac{bh'}{S_i} h_{31} r_{x'} z_{x'} , \tag{2}
\]
where \( z_i' \) is the slope of the beam at the location of the \( i^{th} \) patch.

Using central-difference formula, the deflection at the \( i^{th} \) patch location can be calculated as follows:

\[
  z_i \approx z_i'(x_i - x_{i-1}) + z_{i-1}.
\]  

Equation (3) represents the general form of the beam lateral displacement equation and is independent of the beam boundary conditions; however \( z_0 \) must be known. Therefore the requirement on this method is that the displacement of at least one end of the beam must be known. For clamped and simply supported boundary conditions, the vector \( \{z\} \) of \( z_i \) represents an approximation of the curve shown in Fig. 1. For any other boundary conditions, a point sensor near the origin can be used to find \( z_0 \).

### 2.2 Plate Displacement Sensor Equation

For the vibrating plate displacement sensor, multiple patches of film are used as shown in Fig. 2 with \( p \times q \) patches of films. The \( ij \) patch output charge can be written as:

\[
  \phi_{ij} = -\frac{h_{ij}}{S_{ij}} \int_{x_{j-1,y_{j-1}}}^{y} \int_{x_{j,y_{j}}}^{y} \left[ h_{21} r_x \frac{\partial^2 z}{\partial x^2} + h_{32} r_y \frac{\partial^2 z}{\partial y^2} \right] dxdy.
\]  

The charge \( \phi_{ij} \) can be calculated by separating the equation into two:

\[
  \phi_{ij}^x = -\frac{h_{ij}}{S_{ij}} \int_{x_{j-1,y_{j-1}}}^{y} \int_{x_{j,y_{j}}}^{y} h_{21} r_x \frac{\partial^2 z}{\partial x^2} dxdy,
\]

\[
  \phi_{ij}^y = -\frac{h_{ij}}{S_{ij}} \int_{x_{j-1,y_{j-1}}}^{y} \int_{x_{j,y_{j}}}^{y} h_{32} r_y \frac{\partial^2 z}{\partial y^2} dxdy.
\]
The integration of Eq. (5) along \( x \)-direction and Eq. (6) along the \( y \)-direction yields the following equations:

\[
\phi^x_{ij} = -\frac{h^f_y}{S^f_y} \int_{y_{j-1}}^{y_j} h_{31} r'_{x} \left[ \frac{\partial z}{\partial x} \right]_{x=1}^{x_{i-1}} dy,
\]

\[
\phi^y_{ij} = -\frac{h^f_x}{S^f_x} \int_{x_{j-1}}^{x_j} h_{32} r'_{y} \left[ \frac{\partial z}{\partial y} \right]_{y=1}^{y_{i-1}} dx.
\]

The slopes \( z^x_x = \partial z / \partial x \) in the \( x \)-direction and \( z^y_y = \partial z / \partial y \) in the \( y \)-direction are respectively assumed constant at the location of the \( i \)-th patch. These equation are similar in form to Eq. (1) but their dependency on the gradient in the \( y \)-direction complicates the integration. In practice sensor strip laid as the blue area shown in Fig. 2 where the sensor \( 1 \)-axis is lined up with the \( x \)-axis will output charges that account more for strains in the \( x \)-direction than in the \( y \)-direction. Similar observation can be made for the green filmstrip.

Now using Raleigh formulation, the mode shapes of a plate can be written as the product of beam functions:

\[
W(x, y) = X(x)Y(y),
\]

where \( X(x) \) and \( Y(y) \) are chosen as the fundamental mode shapes of beams having the boundary conditions of the plate. This formulation works well for all plates boundary condition except for free edges where approximate solution is required. Therefore the following discussion will focus on plates with clamped or simply supported edges. For the blue and green sensor strips of Fig. 2, we can derive respectively Eqs. 10 and 11:
Based on the beam sensor theory, the plate deflection along the PVDF strips can be written as

\[
\phi_i^x = -\frac{h_i^f \Delta y}{S_i} h_{s1_i} r_{x_i} z_i^y
\]

(10)

\[
\phi_i^y = -\frac{h_i^f \Delta x}{S_i} h_{s1_i} r_{x_i} z_i^y
\]

(11)

Based on the beam sensor theory, the plate deflection \( z_i^x \) and \( z_i^y \) along the PVDF strips can be written as

\[
z_i^x \equiv z_i^y (x_i - x_{i-1}) + z_{i-1}^x
\]

(12)

\[
z_i^y \equiv z_i^y (y_i - y_{i-1}) + z_{i-1}^y
\]

(13)

where \( z_i^y \) and \( z_i^y \) are respectively calculated from \( \phi_i^x \) and \( \phi_i^y \). From Raleigh formulation, the deflection of the plate is given by the following equation.

\[
z(x, y) \approx z^x z^y
\]

(14)

Equation 14 is only valid when the plate is vibrating at one of its fundamental frequencies because at resonance, the shape of the plate along the \( x \)-direction remains constant across the \( y \)-direction while the shape of the plate along the \( y \)-direction remains constant across the \( x \)-direction. For an off resonance vibration of the plate however, these two sensor strips will only capture the deflections at their locations and Eq. (14) cannot be used.

For plate vibrating off resonance, multiple blue and green film strips can be used such that there will be a top and bottom patch at each \( ij \) location. Therefore at a given \( ij \) location we can measure the slope in the \( x \)-direction as \( z_i^y \) and the slope \( z_i^y \) in the \( y \)-direction. From these two measured slopes at \( ij \) we can calculate the mixed partial derivative \( z_{xy}^{ij} \) as \( z_{xy}^{ij} z_{xy}^{ij} \). The deflection \( z_{xy}^{ij} \) at the location \( ij \) can then be calculated using the central difference equation for mixed partial derivative as follows:

\[
z_{xy}^{ij} (x_{i,j}, y_{i,j}) \approx \frac{1}{2\Delta x \Delta y} (z_{i+1,j} - z_{i-1,j} + z_{i,j+1} - z_{i,j-1})
\]

(15)

The indices begin at 1 and the equation is written such that row \( i = 1 \) corresponds to \( y = 0 \) and column \( j = 1 \) correspond to \( x = 0 \) in the \( x-y \) coordinates system. Therefore, from Eq. (5), it is clear that if the boundary conditions \( z_{i,1} \) and \( z_{i,1} \) respectively along the \( x-axis \) and \( y-axis \) are known the deflection \( z_{xy} \) at \( ij \) can be calculated. The proposed PVDF film arrangement can be used to effectively measure the lateral vibration displacement curve of a plate. This arrangement calls for fully covering the plate with bands of film in the \( x \) and \( y \) direction.

### 3. Numerical simulation

The validity of Eqs. (14) and (15) is tested using numerical simulation with MATLAB. The data of Table 1 is used in the simulation. For the case of the plate vibration at one of the fundamental frequencies, we arbitrary selected modes \((1,1),(1,2),(2,1), \& (2,2)\) for the mode shape couple \((n,m)\) to calculate the “actual” deflections of the plate using Eq. (16) for the simply supported plate and Eq. (17) for the clamped plate.
Table 1. Geometry and physical constants for the plate and PVDF Film.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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<td>Thickness</td>
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</tr>
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<td>Width</td>
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<td></td>
<td>...</td>
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</tr>
</tbody>
</table>

\[
z(x, y) = C_{nm} \left[ \sin(\beta_n x) \right] \left[ \sin(\beta_m y) \right] \quad \text{with} \quad \sin \beta_n L_x = 0 \quad \text{and} \quad \sin \beta_n L_y = 0,
\]

\[
z(x, y) = C_{nm} \left[ \sinh \beta_n x - \sin \beta_n x + \alpha_n \left( \cosh \beta_n x - \cos \beta_n x \right) \right] \times
\]

\[
\left[ \sinh \beta_m y - \sin \beta_m y + \alpha_m \left( \cosh \beta_m y - \cos \beta_m y \right) \right]
\]

\[
\quad \cos \beta_n L_y \cos \beta_m L_y = 1 \quad \text{and} \quad \alpha_n = \frac{\sinh \beta_n L_x - \sin \beta_n L_x}{\cos \beta_n L_x - \cosh \beta_n L_x},
\]

\[
\cos \beta_m L_y \cos \beta_m L_y = 1 \quad \text{and} \quad \alpha_m = \frac{\sinh \beta_m L_y - \sin \beta_m L_y}{\cos \beta_m L_y - \cosh \beta_m L_y}
\]

where \( C_{nm} \) is an amplitude factor.

Figure 3: Plate (SSSS) vibrating in modes (1,1), (1,2), (2,1), & (2,2): Red (“Actual”), Black (“Measured”)

Then using the blue and green strips shown on Fig. 2, the charges \( \phi_i^x \) and \( \phi_i^y \) are calculated from Eqs. (7) and (8) before the deflections \( z_{ij}^x \) and \( z_{ij}^y \) are calculated using Eqs. (12) and (13). The “measured” deflections of the plate are calculated from Eq. (14). Figures 3 and 4 show the plots of the “Actual” and “Measured” deflections. The results indicate that PVDF sensor strips can be used to validate the
Raleigh formulation of the mode shapes of plates and the measurement of deflection curve theory presented for plate vibrating at resonance.

For off resonance vibration of the plate, multiple PVDF strips were used and the slopes in the $x$–direction $z_x$ and $y$–direction $z_y$ were calculated using the same procedure as for the case of plates vibrating at resonance but the deflection curves were calculated from Eq. (15). Once again the results of Fig. 5 indicates that the proposed sensor can be utilized to effectively measure the vibration displacement curve of plates.

![Figure 4: Plate (CCCC) vibrating in modes (1,1), (1,2), (2,1), & (2,2): Red (“Actual”), Black (“Measured”)](image)

![Figure 5: Plates vibrating off resonance: Red (“Actual”), Black (“Measured”)](image)

### 4. Conclusion

We derived formulas for plates lateral vibration displacement sensor using well-established Poly-Vinylidene Fluoride or PVDF output charge equations. The displacements of plates vibrating at resonance can be measured with two perpendicular constant shape films running each respectively along the $x$– and $y$– axis. These sensors are based on the Raleigh formulation of plate mode shapes. For plate vibrating off resonance, multiple bands of film are needed to cover the entire surface of the plate. It is shown that the sensor films actually measure deflection slopes and that central difference equations are used to calculate the deflections.

The proposed sensors were verified through numerical simulation for simply supported and clamped plates. At and off resonance displacements calculated from the sensor outputs were compared to the displacements calculated from plate theory. The results indicate that the proposed PVDF
film sensors can be effectively used to measure the vibration displacement curve of plates with various boundary conditions. Future work will include experimental verifications.

REFERENCES


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