FORCED VIBRATION RESPONSE CHARACTERISTICS OF PERIODIC RECTANGULAR PLATE STRUCTURES BASED ON THE DYNAMIC STIFFNESS METHOD

Chunyu Zhang, Guoyong Jin*, Tiangui Ye, Chuanmeng Yang.

College of Power and Energy Engineering, Harbin Engineering University, Harbin, PR China
*corresponding author, email: guoyongjin/hrbeu.edu.cn

Forced vibration response characteristics of periodic rectangular plate structures with general boundary conditions are studied by using the dynamic stiffness method. Geometrical property is assumed to vary periodically in the whole plate structure. Both of the dynamic stiffness matrices of in-plane and out-of-plane vibration of the unit plate are derived. Once the dynamic stiffness matrix of the unit plate is obtained, the dynamic stiffness matrix of the whole periodic structure will be assembled in a similar manner with the finite element method. Harmonic responses are calculated to investigate the band gap behaviours of the periodic plate structures. Parametric studies are implemented to investigate the effects of the geometrical parameters on the vibration band gap characteristics.

Keywords: dynamic stiffness method, band gap, harmonic response, periodic structure.

1. Introduction

The propagation of elastic waves in artificial periodic media known as phononic crystals (PCs) has received much attention over the years. Many physical properties of PCs have been discovered and investigated. Among them, the band gap property of the periodic structure is one of the most distinct one. When elastic wave propagates in periodic structure, it will be reflected back and forth in the unit cell when it propagates through the interfaces of the adjacent cells and dissipated gradually in certain frequency domain which is defined as band gap domain. The band gap property of periodic structure has great application prospects in engineering field such as, wave barriers, frequency filters, noise control, etc.

Over the past decades, the band gap properties in periodic structures have got wide attention and rapid development, and considerable research achievements have emerged. In engineering, the research subjects were mainly focused on periodic one-dimensional rod and beam, and two-dimensional plate and shell structure. Diaz-de-Anda et.al [1] investigated the flexural vibrations of a locally periodic rod both from the experimental and theoretical points of view. Richard and Pines [2] provided a passive approach to reduce transmitted vibration generated by gear mesh contact dynamics by using a periodic shaft. Xiao et al. [3] studied the flexural wave propagation in locally resonant beams with multiple periodic arrays of attached spring-mass resonators, and results demonstrated that a locally resonant beam with multiple arrays of damped resonators can achieve much broader band gaps. Gei [4] investigated the filtering properties of multi-supported periodic rods and beams with elementary cells constructed following the Fibonacci sequence. By using Floquet theory and boundary integral equations, Sorokin and Ershova [5] studied the steady-state free vibration of non-uniform elastic cylindrical shells with and without internal heavy fluid loading. Søe-Knudsen and Sorokin [6] presented the modelling of wave propagation phenomena in curved pipes individually and in their combinations with other curved and straight elements, including periodicity effects.
The structures of periodically placed plates are also commonly encountered in various practical engineering. In early years, Sigalas [7] investigated the elastic waves propagating in both thick and thin plates consisting of solid inclusions placed periodically in the host material based on the plane wave expansion method. Langley and Smith [8] studied of the effects of disorder in the stiffener spacing on the high frequency vibration transmission through a periodic plates system within the application of statistical energy analysis. Recent years, Wu et al. [9,10] have applied the spectral element method to investigate the dynamic behavior of periodic plate structures with classic plate theory and Mindlin plate theory. Claeyts et al. [11] applied the band gap property to decrease the vibrational response of periodic panels based on Bloch’s theorem.

The purpose of the present work is to present the forced vibration analysis of the periodic plate structures and investigate the band gap behaviours by using the improved DSM [12]. In this paper, the equation of motion of the periodic plate structure is derived. The forced vibration responses are calculated and the vibration band gaps properties of periodic plate structure are investigated. Much analysis is applied to study the effects of geometric properties on the characteristics of the band gaps.

2. Theoretical modelling

![Figure 1](image1)

Figure 1: A periodic plate structure and the corresponding unit cell.

Figure 1 shows the schematic diagram of a periodic plate structure and the corresponding unit cell. The periodic structure is consisted of n end-to-end I shaped cells, and the I shaped cell is made up of 5 individual rectangular plates. The dimensions of the individual plate are assumed as 2a×2b×h. In the formulation, the plate is supposed to be homogeneous, isotropic and elastic and its thickness is uniform and quite small compared to the other dimensions so that the Kirchhoff thin plate theory is applicable.

2.1 Fundamental theory

![Figure 2](image2)

Figure 2: Coordinate system and notations for displacements and forces of the rectangular plate.

Figure 2 shows the displacements, forces and moments along boundary edges in the local coordinate system. The notations u, v and w represent the displacement fields and \(V_{xy}, N_{xy}, N_y\) and \(M_{xy}\) represent the transverse force, normal force, tangential force and bending moment, respectively.
Based on the thin plate theory, the expressions of rotations, forces and moments can be given as follows.

- **Out-of-plane:**

  \[
  \phi_x = - \frac{\partial w}{\partial x}, \phi_y = - \frac{\partial w}{\partial y} \quad (1)
  \]

  \[
  M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (2)
  \]

  \[
  M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (3)
  \]

  \[
  V_x = -D \left[ \frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] \quad (4)
  \]

  \[
  V_y = -D \left[ \frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right] \quad (5)
  \]

  where \( D = \frac{Eh^3}{12(1-\nu^2)} \).

- **In-plane:**

  \[
  N_x = \frac{Eh}{1-\mu^2} \left( \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) \quad (6)
  \]

  \[
  N_y = \frac{Eh}{1-\mu^2} \left( \nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad (7)
  \]

  \[
  N_{xy} = \frac{Eh}{2(1+\nu)} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = N_{yx} \quad (8)
  \]

In the equations, \( D, E \) and \( \nu \) are the flexural rigidity, Young’s modulus and Poisson’s ratio, respectively.

Introducing spectral representation of the displacements, and as:

\[
\hat{u}(x, y, t) = \hat{u}(x, y, \omega) e^{i\omega t} \quad (9)
\]

where \( \hat{u}(x, y, \omega) \) is the amplitude of the displacement filed \((u, v, w)\) in the frequency domain. Therefore, the frequency-dependent governing equations for out-of-plane and in-plane motions are given as:

\[
\frac{\partial^4 \hat{w}}{\partial x^4} + 2 \frac{\partial^4 \hat{w}}{\partial x^2 \partial y^2} + \frac{\partial^4 \hat{w}}{\partial y^4} + \rho \omega^2 \hat{w} = \hat{f}_w \quad (10)
\]

\[
\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{1+\nu}{2} \frac{\partial^2 \hat{v}}{\partial x \partial y} - \frac{(1-\nu^2)\rho \omega^2}{E} \hat{u} = \hat{f}_u \quad (11)
\]

\[
\frac{\partial^2 \hat{v}}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 \hat{v}}{\partial y^2} + \frac{1+\nu}{2} \frac{\partial^2 \hat{u}}{\partial x \partial y} - \frac{(1-\nu^2)\rho \omega^2}{E} \hat{v} = \hat{f}_v \quad (12)
\]

where \( \rho \) is the mass density and \( \omega \) is the angular frequency. \( \hat{f}_w, \hat{f}_u \) and \( \hat{f}_v \) are the external forces in the \( z, x, \) and \( y \) directions, respectively. As one can see in Eqs. (10-12), the out-of-plane and in-plane motions are decoupled, therefore the dynamic stiffness matrices for these two cases should be formulated separately.
2.2 Formulation of the dynamic stiffness matrix of rectangular plate

Development of the dynamic stiffness matrix for rectangular plate element undergoing transverse and in-plane vibrations have been given in detail in [12]. In this part, only the main strategy of the derivational process is presented. Based on the superposition method, arbitrary displacement field can be split into four contributions according to the symmetry/antisymmetric properties of the rectangular plate, which are symmetric-symmetric (SS), symmetric-antisymmetric (SA), antisymmetric-symmetric (AS) and antisymmetric-antisymmetric (AA) as:

\[ \tilde{u}(x, y, \omega) = \tilde{u}^{SS}(x, y, \omega) + \tilde{u}^{SA}(x, y, \omega) + \tilde{u}^{AS}(x, y, \omega) + \tilde{u}^{AA}(x, y, \omega) \]  

The general solution of the equation of motion can be defined in a concise form as:

\[ \tilde{u}^{ij}(x, y, \omega) = \sum_{m=0,1}^{M} C_{m}^{ij} f_{m}^{ij}(x) g_{m}^{ij}(y) \]  

where, \( f_{m}^{ij}(x) \) and \( g_{m}^{ij}(y) \) are basis trigonometric functions which depend both on symmetry property and the solution of corresponding governing equation, \( C_{m}^{ij} \) is the undetermined coefficient matrix, \( M \) is the truncated number of the series, while \( i, j = S, A \).

Then, by substituting \( \tilde{u}^{ij} \) into the force-displacement relation expressions in Eqs. (1-8), the force vector can be obtained as:

\[ \hat{f}^{ij}(x, y, \omega) = \sum_{m=0,1}^{M} C_{m}^{ij} f_{m}^{ij}(x) g_{m}^{ij}(y) \]  

where, the trigonometric functions \( f_{m}^{ij}(x) \) and \( g_{m}^{ij}(y) \) can be deduced accordingly.

This method allows one to analyse only one quarter of the rectangular plate. Therefore, the displacement and force vectors on the plate edges can be written in a vector form as:

\[ \tilde{q}^{ij} = \begin{bmatrix} \tilde{u}_{x=0}^{ij} \\ \tilde{u}_{y=0}^{ij} \end{bmatrix}, \quad \hat{Q}^{ij} = \begin{bmatrix} \hat{f}_{x=0}^{ij} \\ \hat{f}_{y=0}^{ij} \end{bmatrix} \]  

The projection method is introduced to establish the relation between the spatial dependent displacement and force along the boundary edges. The projections of the displacement and force vector are defined as

\[ \bar{q}^{ij} = \frac{2}{L} \int_{s} H^{ij} \hat{q}^{ij} ds = D^{ij} \bar{C}^{ij} \]  

\[ \bar{Q}^{ij} = \frac{2}{L} \int_{s} H^{ij} \hat{Q}^{ij} ds = F^{ij} \bar{C}^{ij} \]  

where \( H^{ij} \) is the matrix of the projection functions. By eliminating the vector \( C^{ij} \) in Eq. (17) and (18), the DS matrix of the quarter of the plate \( \tilde{K}^{ij} \) is obtained as:

\[ \bar{Q}^{ij} = F^{ij} \left(D^{ij} C^{ij}\right)^{-1} \bar{q}^{ij} = \tilde{K}^{ij} \bar{q}^{ij} \]  

Then the DS matrix \( \tilde{K} \) for the completely rectangular plate can be obtained by corresponding elementary transformation, and detailed procedure can be referred to Ref. [13].

2.3 Dynamic stiffness matrix for the periodic plate structure

Once the DS matrix of the individual plate has been deduced, rotated and offset if required, it can be assembled into a whole DS matrix. The procedure is similar to that used in the FEM, except that the plates are connected along the boundary condition instead of nodes. For illustrative purpose, the assemble procedure of the DS matrix of a coupled structure is schematically shown in Fig. 3. Consequently, the equation of motion of the whole periodic plate structure can be derived as

\[ \tilde{K} \bar{q} = \hat{f} \]
where $\bar{K}_w$ is the global dynamic stiffness matrix of the periodic plate structure, and $\bar{d}_w$ and $\bar{f}_w$ are the global displacement and force vectors. With regard to the clamped and simply supported boundary, there are zero elements in the vector $\bar{d}_w$ which will be removed and the corresponding rows and columns of $\bar{K}_w$ will be condensed. Note that, Eq. (20) is derived in frequency domain, therefore the harmonic response can be obtained by solving Eq. (20), and the vibration band gap behaviours can be studied directly.

3. Results and discussions

In this section, the vibration band gap properties of the periodic plate structure shown in Fig. 1 are investigated. All the sub-plates are assumed to be homogeneous, isotropic, and of uniform geometric dimension without special instructions. The plate parameters are taken as $a = b = 0.5$ m, $h = 0.005$ m, $E = 207$ GPa, $v = 0.3$ and $\rho = 7850$ kg/m$^3$. The unit cell number is $n = 7$. In the following analysis, the external excitation $F = F_0 e^{i\omega t}$ is located at middle position of the left edge and the response point is located at the right edge in the periodic plate, as shown in Fig.1. For the sake of space, only one type of boundary condition (all the boundary lines parallel to $x$ and $z$ axes are clamped and the boundary lines parallel to $y$ axis are free) is adopted.

3.1 Convergence and validation

![Figure 3: Assembly of dynamic stiffness matrices.](image)

To validate the convergence and accuracy of the present method, the present results obtained with different truncated number $M$ are given in Fig. 4. For comparison, results obtained by the FEM are also given. The FEM is performed by the ANSYS software using shell 63 element, and each individual rectangular plate is divided into 24×24 elements. From the results, it can be observed that even when $M = 2$ the DSM results show good convergence in frequency range from 0-1000 Hz. Within frequency from 1000-2000 Hz, the results of $M = 3$ and $M = 4$ are nearly consistent. In addition, the results obtained by the DSM with $M = 3$ show an excellent agreement with the FEM results. Therefore,
it can be indicated that the DSM results have converged within the frequency between 0 and 2000 Hz for \( M = 3 \), and the results are accuracy and reliable by comparing with the FEM results.

Several distinct band gaps can be observed in the response curves (i.e., 118-142Hz, 194-270Hz, 378-507Hz, 667-765Hz, 1132-1280Hz, 1527-1653Hz). The vibration amplitudes in the stop bands are much smaller than those in the pass bands. For further illustration, Fig. 5 shows the responses of 7 points which are distributed left-to-right in the periodic structure for frequencies located in the pass bands \((f = 150Hz, f = 350Hz and f = 508Hz)\) and stop bands \((f = 240Hz, f = 750Hz and f = 1200Hz)\). Obviously, the responses show strong attenuation when the frequencies are located in the stop bands as the response point moves away from the excitation point. On the contrary, when the frequencies are located in the pass bands, the responses amplitudes are nearly undiminished. From above, it can be concluded that the elastic waves and vibration can hardly propagate through the periodic structure in the band gap frequencies.

![Figure 5: Vibration responses for 7 points distributed left-to-right in the periodic structure for frequency located in the pass bands \((f = 150Hz, f = 350Hz and f = 508Hz)\) and stop bands \((f = 240Hz, f = 750Hz and f = 1200Hz)\).](image)

3.2 Parametric study

Figure 6 shows the comparison of frequency responses of the periodic structures consist of 4, 7 and 10 unit cells. As one can see that, the locations and widths of the stop bands remain unchanged as the number of the unit cells increases. The change is mainly reflected on the depths of the band gaps, the more cells in the periodic structure the deeper of the gaps. Therefore, in the stop bands, the propagation of wave and vibration in the periodic structure becomes much harder when the it contains more unit cells.

![Figure 6: Comparison of frequency responses of periodic structure with different unit cells number](image)

Next, the effects of the thickness of the unit plate on the band gap behaviours is studied. In Fig. 7(a), the frequency responses of the periodic structure with different thicknesses of the unit plate is depicted. The thicknesses are selected as \( h_1 = 0.005m \), \( h_2 = 0.007m \) and \( h_3 = 0.009m \). Firstly, the
resonance peaks move left because of the increase of the thickness enlarges the structural stiffness. Meanwhile, the locations and widths of the stop bands also change obviously. Figure 7(b) shows the variations of the first four stop bands with respect to the thickness. In Fig. 7(b), the colour shaded parts represent the stop band of the vibration response and the corresponding colourless parts indicate the pass bands. When the thickness increases from $h_1$ to $h_3$, the bandwidth of the colour shaded part becomes larger. What’s more, the relative frequency ranges of the stop bands increase obviously as the thickness increases. In conclusion, the thickness of the unit plate not only effects the bandwidth of the band gaps, but also effects the frequency range of band gaps evidently.

![Figure 7](image)

**Figure 7:** (a) Comparison of frequency responses of periodic structure with different thicknesses of the unit plate, (b) Variation of the frequency ranges of the first four stop bands of the periodic plate.

4. Conclusions

In this paper, the wave propagation and vibration transmission in the periodic plate structures have been investigated. A well-developed strong-form based dynamic stiffness method is applied to predict the frequency responses. The convergence and accuracy of the present method are validated by comparing with the FEM. From the frequency responses, the band gap behaviours can be observed evidently. In the stop bands, the propagation of elastic wave and vibration in the periodic structure are forbade, but, no restraint in the pass bands. In addition, the more unit cells in the periodic structure the deeper the band gaps are, but the bandwidth and location of the stop band is nearly unchanged. The increase of the thickness of the unit plate can not only broaden the bandwidths of the stop bands but also increase the frequency ranges of the stop bands. Consequently, the location and shape can be manipulated optional by adjust the geometrical property of the periodic plate structure.

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