A RECURSIVE METHOD FOR FORCE LOCALIZATION AND RECONSTRUCTION

Qiaofeng Li* and Qiuhai Lu
School of Aerospace Engineering, Tsinghua University, Beijing 100086, China
∗email: liqiaofeng1990@yeah.net

The knowledge of exciting forces is desired for design and health monitoring of structures. Traditional time domain force reconstruction techniques require the exact force location, which restricts their application in engineering practice. In this paper, a recursive estimation method is proposed to address the problem. The proposed method has the basic structure of a Kalman filter. The system states and input forces are jointly estimated. For better reconstruction performance, the prior information on spatial distribution of exciting forces is considered at each step. The method is validated by simulations of a beam structure. With prior information on spatial-sparsity, the forces could be adaptively localized and reconstructed in a pseudo on-line manner.

Keywords: force reconstruction, recursive estimation, spatial sparsity, Kalman filter

1. Introduction

Force reconstruction refers to the problem of reconstructing the histories of forces imposed on a structure, based on measured structural responses. The reconstructed force histories may contain information crucial to design and health monitoring of structures. So in response to popular engineering demands, force reconstruction or identification has become a heated topic for the past decades.

Most of force reconstruction researches assume the location of external forces a prior known. In such cases, the system model (the force-response mapping model) could be constructed analytically, numerically or experimentally by well-established methods, especially for time-invariant linear elastic structures. Then force reconstruction mainly refers to the optimization of force history. Most attention is paid to the development of techniques to overcome the intrinsic ill-posedness of the problem. Jacquelin et al. [1] applied Tikhonov regularization and Truncated Singular Value Decomposition to filter out trivial singular values of the system matrix, which amplifies the effect of measurement noise during system inversion. Liu et al. [2] applied moving least squares fitting to approximate the unknown force with a few smooth functions in time-concentrated support domains. The condition number of the reformed system matrix is effectively improved and the optimization problem becomes less ill-posed. Qiao et al. [3, 4] have considered $\ell_1$ regularization to well-pose the problem. Several solution algorithms are proposed, such as sparse reconstruction by separable approximation (SpaRSA) and primal-dual interior point method (PDIPM). Results have shown that good reconstruction performance could be achieved with these algorithms, as long as the force has a sparse profile or could be sparse-represented in a chosen basis.

In engineering practice, the force locations are sometimes also to-be-inferred, which makes the above methods useless. Several methods have been proposed to simultaneously localize the force and reconstruct the force history, referred as the force localization and reconstruction problem here. Li...
and Lu [5] considered the system matrix as a function of the force location. The optimal location and history of the force is solved by minimizing the object function, with complex method in a two-step iterative approach. Qiao et al. [7] considered localization and reconstruction of impacts from several potential locations. \( l_1 \) regularization is adopted to promote sparsity in the solution. Results show that the force vector corresponding to the impact location could be well-reconstructed, and that the force vectors corresponding to non-impact locations remain zero. Rezayat et al. [8] applied \( l_1 \& l_2 \)-norm regularization to localize and reconstruct forces from a few possible locations. With the layered regularization, the reconstruct-able force is not restricted to impacts but rather arbitrary. Aucejo and De Smet [9, 10, 11] have developed a series of work based on Bayesian formulation. The reconstruction is performed in frequency domain, and regularization is imposed on spatial distribution of forces at each frequency. They divided a plane structure into boundary and non-boundary areas, and applied \( l_2 \) and \( l_1 \) regularization to respective areas. The frequency spectral of both the boundary forces and an impact in the non-boundary area could be reconstructed. The impact could also be localized. Faure et al. [12] also considered Bayesian reconstruction of vibration sources. The posterior force distribution of a structure at each frequency is generated through Markov chain monte carlo (MCMC) method. The posterior distribution indicates the possibility of force locations and amplitudes.

However, notice that although these methods exhibit good localization ability, they are all batch methods. The localization and reconstruction could only be performed when enough measurements are collected. In other words, they are off-line approaches. Off-line approaches generally consume more computation resources than on-line iterative approaches, such as Kalman filter. They may also encounter difficulties when the force location is fast changing or multiple forces are happening consecutively at different locations. Kalman filter [13, 14, 15] has been adopted in the context of force reconstruction. Notice the previous applied Kalman filters [13, 14, 15] assume the force locations a priori known, thus are not suitable for localization and reconstruction tasks. To provide Kalman filter with the localization ability, we need to borrow the sparsity-inducing idea. The result is called Sparse Kalman filter (SKF).

In this paper, we propose the SKF for a general state-space dynamic model. The SKF follows the standard Kalman filter prediction-update manner. While different from classic Kalman filters, at each step, the system input is predicted using relevance vector machine (RVM) [16, 17]. RVM requires no user-defined parameters besides promoting sparsity in the solution. So the requirement of expert knowledge is reduced to the least and the robustness of the method is supposed to be guaranteed. More importantly, with the sparsity promoting ability, SKF could accurately localize and reconstruct forces when observability is violated in the traditional sense. This property is considered a large step forward compared to classic methods when in engineering practice only limited sensors can be used.

The rest of the paper is organized as follows. Section 2 introduces the proposed SKF. The performance of SKF is illustrated in Section 3 via numerical simulations. Section 4 concludes this research.

2. Sparse Kalman Filter

For an arbitrary linear elastic structure, we consider modelling the force-response relationship with the following discrete state-space model,

\[
\begin{align*}
x_k &= A x_{k-1} + B f_{k-1} \\
y_k &= C x_k + v_k
\end{align*}
\]  

where \( x_k \in \mathbb{R}^{n \times 1} \) and \( v_k \in \mathbb{R}^{p \times 1} \) denote respectively the state vector and measurement noise vector at the \( k \)th time instant, \( f_{k-1} \in \mathbb{R}^{m \times 1} \) is the force vector at the \((k - 1)\)th time instant. The measurement noise \( v_k \) is assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrix \( R = R_k = \mathbb{E}[v_k v_k^T] \). \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{p \times n} \) are state-space model matrices. Such matrices could be constructed analytically, numerically (e.g. refer to [18]) or experimentally, which is not detailed in this paper. In this model, we consider \( p \) sensors and \( m \) potential
force locations. The real forces are imposed at one or some of these locations. The overall task is to localize and reconstruct forces from \( m \) potential locations based structural response measurements collected from \( p \) sensors.

We consider a recursive filter of the form

\[
\begin{align*}
\hat{x}_{k|k-1} &= A\hat{x}_{k-1|k-1} \\
\hat{f}_{k-1} &= M_k(y_k - C\hat{x}_{k|k-1}) \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + B\hat{f}_{k-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k} + K_k(y_k - C\hat{x}_{k|k})
\end{align*}
\]

(3) (4) (5) (6)

where \( M_k \in \mathbb{R}^{m \times p} \) and \( K_k \in \mathbb{R}^{n \times p} \) are to-be-determined. This filter form is almost identical to that in [19]. However, due to the differences in solution, \( M_k \) and \( K_k \) will be greatly different.

2.1 Input estimation

The input estimation is performed with RVM [16, 17], and is only briefly introduced here. On the derivation and sparsity-inducing mechanism of RVM, please refer to [16, 17]. In this section, we denote \( y_k - CX_{k|k-1} \) as \( t \), and \( f_{k-1} \) as \( f \) for simplicity. From Eqs. (1), (2) and (3),

\[
t = y_k - C\hat{x}_{k|k-1} = CBf + CA\hat{x}_{k-1|k-1} + v_k
\]

(7)

where \( \hat{x}_{k-1|k-1} = x_{k-1} - \hat{x}_{k-1|k-1} \). Define \( \Phi = CB \). Based on the well-known Bayes’ rule, if we consider \( t \) as a random vector given a certain \( f \), the conditioned probability of \( t \) is

\[
p(t|f) \propto \exp \left( -\frac{1}{2} (\Phi f - t)^T \Sigma^{-1} (\Phi f - t) \right)
\]

(8)

where \( \Sigma = (CA)P_{k-1}(CA)^T + R \); \( P_{k-1} \triangleq \mathbb{E}(\hat{x}_{k-1|k-1}X_{k-1|k-1}^T) \) is state estimation error covariance matrix at the \((k-1)\)th step. Consider the singular value decomposition (SVD) of \( \Sigma \), i.e. \( \Sigma = UT^SU \) (note \( \Sigma \) is positive definite). \( S \) is a diagonal matrix containing the singular values of \( \Sigma \), i.e. \( S = \text{diag}(s_1, s_2, \cdots, s_p) \). We can make the transformation

\[
p(t|f) \propto \exp \left( -\frac{1}{2} (\Phi f - t)^T \Sigma^{-1} (\Phi f - t) \right) \\
= \exp \left( -\frac{1}{2} (\Phi f - t)^T U^T S^{-1} U (\Phi f - t) \right) \\
= \exp \left( -\frac{1}{2} (\Phi \bar{f} - \bar{t})^T (\Phi \bar{f} - \bar{t}) \right)
\]

(9)

Corresponding matrices are

\[
\overline{\Phi} = T\Phi \quad \overline{t} = Tt \quad T = S^{-\frac{1}{2}}U
\]

(10)

We assume that \( f \) has the prior distribution of the form

\[
p(f|\alpha) \propto \prod_{i=1}^{m} \exp \left( -\frac{\alpha_i f_i^2}{2} \right)
\]

(11)

where \( f_i \) and \( \alpha_i \) are the \( i \)th element of \( f \) and \( \alpha \in \mathbb{R}^{m \times 1} \) respectively. That is to say, we assume that the elements of \( f \) are independently distributed, and each is assigned a zero-mean Gaussian distribution with variance parameter \( \alpha_i^{-1} \). The posterior distribution of \( f \) could be derived with the Bayes’ rule,

\[
p(f|t, \alpha) = p(t|f)p(f|\alpha)/p(t|\alpha)
\]

(12)
and is Gaussian $\mathcal{N}(\mu, \Sigma_1)$ with
\[
\Sigma_1 = \left( A_1 + \Phi^T \Phi \right)^{-1} \quad \mu = \Sigma_1 \Phi^T \hat{t}
\] (13)
where $A_1$ is defined as diag($\alpha_1, \alpha_2, \cdots, \alpha_m$). From here, the above derived formulation is in accordance with the standard formulation in [17], in which $\alpha$ is solved by maximizing the marginal likelihood $p(\alpha_i | \alpha)$ with respect to $\alpha$. In the process, no additional parameter settings are needed. Once an optimal $\alpha$ is solved, a most-probable $\alpha_p$ (Eq. (13)) is calculated. Due to page limitations, the solution algorithm is not introduced here. Please refer to [16, 17] for details. Note from Eq. (13), we have
\[
\mu = \Sigma_1 \Phi^T \hat{t} = (A_1 + \Phi^T \Sigma^{-1} \Phi)^{-1} \Phi^T \Sigma^{-1} \hat{t}
\] (14)
So we have $M_k = \left( A_1 + \Phi^T \Sigma^{-1} \Phi \right)^{-1} \Phi^T \Sigma^{-1}$.

2.2 State estimation

Once we have an estimate of $f_{k-1}$ as $\hat{f}_{k-1}$, the next is to derive the Kalman filter gain $K_k$. In this section, we will first derive the error covariance matrix of $\hat{x}_{k|k}$, then the $K_k$ and the error covariance matrix of $\hat{x}_{k|k}$. Defining $\tilde{f}_{k-1} \triangleq f_{k-1} - \hat{f}_{k-1}$, we have
\[
\tilde{f}_{k-1} = f_{k-1} - M_k (CA \hat{x}_{k-1} + v_k + \Phi f_{k-1}) = (I - M_k \Phi) f_{k-1} - M_k CA \hat{x}_{k-1} - M_k v_k
\] (15)
Defining $\hat{x}_{k|k} \triangleq x_k - \hat{x}_{k|k}$, from Eqs. (1) and (5), we have
\[
\hat{x}_{k|k} = A \hat{x}_{k-1} + B \tilde{f}_{k-1} = (I - BM_k C) A \hat{x}_{k-1} + B (I - M_k \Phi) f_{k-1} - BM_k v_k
\] (16)
Considering $\mathbb{E}(\tilde{x}_{k-1} v_k^T) = 0$, $\mathbb{E}(f_{k-1} f_k^T) = 0$ and $\mathbb{E}(\tilde{x}_{k-1} f_{k-1}^T) = 0$ (the last equity is due to the independence of external structural excitations w.r.t system state), the following equation is derived,
\[
P_{k-1}^{x} \triangleq \mathbb{E}(\hat{x}_{k|k} \hat{x}_{k|k}^T) = (I - BM_k C) A \hat{x}_{k-1} A^T (I - BM_k C)^T + B (I - M_k \Phi) P_{k-1}^f (I - M_k \Phi)^T B^T + BM_k R (BM_k)^T
\] (17)
where $P_{k-1}^f$ is the covariance matrix of $f_{k-1}$. Note this covariance matrix is of $f_{k-1}$ instead of $\hat{f}_{k-1}$, thus is corresponding to the prior information of $f_{k-1}$. Recall that in Section 2.1 we assigned prior distribution to $f_{k-1}$ as zero-mean Gaussian distribution with individual covariances $\alpha_i^{-1}$. Although $\alpha$ is then estimated by taking the $i$th step measurement $y_k$ into account, it still serves as a good estimate of the true prior information of $f_{k-1}$. So we take
\[
P_{k-1}^f = A_1^{-1}
\] (18)
From Eq. (6),
\[
\hat{x}_{k|k} \triangleq x_k - \hat{x}_{k|k} = (I - K_k C) \hat{x}_{k|k} - K_k v_k
\] (19)
The error covariance matrix $P_k^x \triangleq \mathbb{E}(\hat{x}_{k|k} \hat{x}_{k|k}^T)$ is then given by
\[
P_k^x = K_k \hat{R}_k K_k^T - V_k^T K_k^T - K_k V_k^T + P_k^{x^*}
\] (20)
where
\[
\hat{R}_k = CP_k^x C^T + R + CS_k^* + S_k^{*T} C^T
\] (21)
\[
V_k^* = P_k^{x^*} C^T + S_k^*
\] (22)
\[
S_k^* \triangleq \mathbb{E}[(x_{k|k} - \hat{x}_{k|k}) v_k^T] = -BM_k R
\] (23)
Minimizing the trace w.r.t $K_k$, yields $K_k = V_k^* (\hat{R}_k)_{xx}^{-1}$ and $P_k^x = P_k^x - K_k V_k^T$.

By far a recursive step is completed. If we start with an estimate of the initial condition of the system $x_{0|0}$ and the error covariance matrix of that estimate $P_{0|x}$, we can estimate $f_0$ according to Eq. (4) based a new response measurement $y_1$. The current state estimate $\hat{x}_1$ and its error covariance matrix $P_{1|x}$ can then be calculated according to Eq. (5) and Eq. (20) respectively. Estimation could continue recursively once a new measurement is obtained. Note due to the lack of force-to-response direct transmission term, the force estimation is always one step delayed w.r.t the response measurement.

3. Numerical Validation

In this section, we are illustrating the proposed SKF via simulation of a beam structure. The beam structure is shown in Fig. 1. It’s clamped at both ends. The Young’s modulus and density of the beam are $2.06\times 10^5$ MPa and $7800$ kg/m³. The length, width and height of the beam are respectively 1m, 3cm and 1cm. A total of 7 displacement sensors are installed at locations $s_1$, $s_2$, ..., $s_7$. Corresponding coordinates of the locations are listed in Table 1. The beam is meshed with 100 Euler-Bernoulli beam elements. System matrices $A$, $B$ and $C$ are constructed numerically. The sampling frequency is set as $1024$Hz. Two forces, $f_1$ and $f_2$ are respectively imposed at $s_4$ and $s_8$. $s_1$ - $s_9$ are all considered potential force locations. Note in this case the number of potential force locations is larger than the number of sensors, so observability is violated in a traditional sense. Classic minimum-variance unbiased (MVU) recursive method [19] thus becomes useless.

![Illustration of the beam structure. Both ends of the beam are clamped. Two forces are imposed at $s_4$ and $s_8$.](image)

Table 1: The $x$ coordinates (m) of the 9 locations.

<table>
<thead>
<tr>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$s_6$</th>
<th>$s_7$</th>
<th>$s_8$</th>
<th>$s_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>0.25</td>
<td>0.38</td>
<td>0.50</td>
<td>0.63</td>
<td>0.75</td>
<td>0.87</td>
<td>0.19</td>
<td>0.81</td>
</tr>
</tbody>
</table>

40dB Gaussian white noise is added to measurements of the sensors. Reconstructed force histories are shown in Fig. 2. It is observed that reconstructed force histories at non-force locations have relatively low amplitudes and energy. Noisy spikes at the non-force locations are quite negligible except for $s_1$ and $s_2$. However, although noisy spikes exist at $s_1$ and $s_2$, we think it’s reasonable to determine them as non-force locations since their reconstructed histories seem quite meaningless. At real-force locations, the trend of true force histories is well captured by the reconstructed force histories. We denote $f_{s_i}$ as the reconstructed force history at $s_i$, and define the following notations

$$\beta_i = ||f_{s_i}||_2$$

$$\gamma_i = \frac{\beta_i}{\sum_{j=1}^9 \beta_j}$$
The reconstruction errors (RE) are defined as

\[
RE = \|F_{\text{recon}} - F_{\text{true}}\|_2
\]  

Figure 2: Reconstructed (blue solid) and true (red solid) force histories of \(s_1 - s_9\). The location is indicated by the number in each subfigure.

<table>
<thead>
<tr>
<th></th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
<th>(s_5)</th>
<th>(s_6)</th>
<th>(s_7)</th>
<th>(s_8)</th>
<th>(s_9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RE</td>
<td>101.99</td>
<td>108.51</td>
<td>54.41</td>
<td>128.29</td>
<td>35.30</td>
<td>25.26</td>
<td>10.30</td>
<td>152.68</td>
<td>11.35</td>
</tr>
<tr>
<td>(\gamma_i)</td>
<td>0.0647</td>
<td>0.0685</td>
<td>0.0149</td>
<td>0.5454</td>
<td>0.0148</td>
<td>0.0038</td>
<td>0.0014</td>
<td>0.2857</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

\(\gamma_i\) is the relative energy percentage of the reconstructed force history at \(s_i\) over all reconstructed histories. It serves as an indicator for localization performance. Since the forces at \(s_4\) and \(s_8\) are in fact constructed by translating the same data sequence, the theoretical \(\gamma_i\)s at \(s_4\) and \(s_8\) are both 0.5. From Table.2, the \(\gamma_i\)s at non-force locations are generally small, which also indicates that these locations are non-force locations. \(\gamma_4\) is close to 0.5, while \(\gamma_8\) has a larger error. It is concluded that force histories at uncollocated locations are harder to identify than at collocated locations.
4. Conclusions

In this paper, we propose the sparse Kalman filter (SKF) for recursive localization and reconstruction of forces imposed on structures. SKF applies relevance vector machine (RVM) for force prediction, instead of MVU estimate as in classic methods. With the sparsity-promoting ability of RVM, SKF could be applied for simultaneous localization and reconstruction of forces. Different from existing methods designed for localization-and-reconstruction tasks, SKF is performed recursively. The computation burden is thus much eased, and the method responds fast to fast change of force locations, as proven in the numerical simulation. Furthermore, SKF allows the violation of observability, i.e. the number of potential force locations could be larger than that of sensors. This property is considered crucial since limited sensors could be installed in practice due to economic considerations. Techniques to improve SKF performance in lower Signal-to-Noise conditions will be developed in future work. The performance of SKF in real-world, unideal conditions will also be tested in the future.

REFERENCES


