In this paper, the doubly clamped microbeam-based resonator which actuated by two electrodes is investigated. A section parameter is proposed to describe the microbeam variables in the upper and under section. The differential equation is derived by considering the influence of neutral surface tension and the nonlinearity of electrostatic force. The Galerkin and Newton Cotes methods are used to obtain reduced-order model. The nonlinear vibration behaviors of system at equilibrium point are analysed. First, the frequency response is obtained by the method of multi-scale (MMS). The relationship between section parameter and mechanical property of the microresonator is studied. Then, the system is optimized by considering the relationship between section parameter and the actual physical parameters. Finally, the finite element method through COMSOL software is used to verify the theoretical results, and the two results are very close to each other in the stable region. The research results provide a good reference for the design of microresonator.

Keywords: microbeam, nonlinear, variable section, optimized

1. Introduction

Microbeam has been widely studied as the main element in micro-electro-mechanical system (MEMS) resonator [1,2]. There are many ways to actuate microresonator, where electrostatic actuation is one of the most common methods because of the advantages of high energy density, fast response, and compatible IC manufacturing process [3]. Moreover, the electrostatic force make microbeams exhibit rich nonlinear behaviors. One of the most common phenomena is pull-in effect [4,5]. The microbeam will quickly pull together with electrode plate when drive voltage exceeds a certain value [6]. The structural will be damaged in serious cases. Therefore, it is very important to improve the mechanical properties of microresonator by increasing pull-in voltage and pull-in location. However, most studies are based on rectangular beams [7]. Recently, scholars pay more attention to variable geometry microbeams [8]. Mechanical properties of resonator can be changed by adjusting the shape of microbeam. This is a very effective method for resonator optimization. Najar et al [9] simulated and analysed the deflection and motion of variable section beams in MEMS devices, and the influence of changing their geometrical parameters on the pull-in location and pull-in voltage was
observed. It is found that adjusting microbeam thickness section shape can significantly increase the pull-in voltage and pull-in location of the resonator. Kuang and Chen [10] investigated the influence of microbeam shape and gap distance on pull-in voltage by considering the neutral surface tension, axial residual stress and electrostatic edge effect. Analytical results demonstrate that the shaped of microbeam with curved electrode can increase the working voltage range approximately six times compared to the rectangular microbeam and flat electrode. Joglekar and Trivedi [11,12] proposed a parametric equation which can smoothly changes the shape of microbeam in width direction. The equation parameters were optimized to increase the pull-in location.

In addition to the pull-in voltage, the change in section also have a very significant effect on microbeam vibration. Najar et al [13] solved dynamic response of variable section beam by differential quadrature method. It is found that the section change have a huge impact on the frequency response curve and the dynamic pull-in effect can even be eliminated by properly adjusting the microresonator geometry. Zhang [14] further discussed dynamic response of the optimized microbeam shape on the basis of reference [11,12]. The results show that vibration amplitude decreases with the width increase of the microbeam middle part.

So far there are few researches on microbeam shape optimization. In particular, the influence of section change on the secondary pull-in phenomenon under two plate actuating has not been reported yet. The doubly clamped microbeam resonator which actuated by two electrodes is considered in this paper. The microresonator is optimized by adjusting microbeam section shape in thickness direction. The influence of section change on the softening and hardening behavior of frequency response is investigated. Then, the physical parameters of microresonator are optimized with linear vibration as the optimization condition. The theoretical results have been verified by finite element software simulation.

2. Mathematical Model

2.1 Governing Equation

The schematic diagram of microbeam is shown in Fig. 1. The model consists of two fixed plates and a movable microbeam. The fixed plates are applied bias voltage $V_{dc}$ and AC voltage $V_{ac} \sin(\Omega t)$.

The thickness of microbeam changes according to $y_1(x) = h/2 + \lambda h \sin \pi x/l$ and $y_2(x) = -h/2 + \lambda h \sin \pi x/l$, where $\lambda$ is defined as a section parameter. The section curvature of microbeam is different with the change of $\lambda$. There are three cases for $\lambda$: $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

The thickness of two clamped sides is greater than the thickness of the middle portion. When $\lambda > 0$, the thickness gradually decreases from the middle to both of the clamped sides. The $\lambda = 0$ case is the ideal case where the section beam thickness is uniform. The cross-sectional area and moment of inertia are $A(x) = A_0(1 + 2\lambda \sin \pi x/l)$ and $I(x) = I_0(1 + 2\lambda \sin \pi x/l)^3$ respectively. The motion equations of microresonator can be written as [15]:

![Figure 1: Schematic of an electrically actuated microbeam](image-url)
\[
\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \right) + \rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} + c \frac{\partial y(x,t)}{\partial t} = \frac{E}{2l^4} \int_0^1 A(x) \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right)^2 dx \frac{\partial^2 y(x,t)}{\partial x^2} ,
\]
(1)

where \( l, b, E, \rho \) are the length, width, effective Young’s modulus and material density, respectively. \( c \) is the damping coefficient. \( \varepsilon_0 \) is the dielectric constant in the free space. \( \varepsilon_r \) is the relative permittivity of the gap space medium with respect to the free space. \( d \) is the distance between plates and the microbeam at two clamped sides.

For convenience, the following nondimensional quantities are defined:
\[
\begin{align*}
\bar{x} &= \frac{x}{l}, \quad \bar{y}(\bar{x}) = \frac{y(x)}{d}, \quad \bar{y}_1(\bar{x}) = \frac{y_1(x)}{d}, \quad \bar{y}_2(\bar{x}) = \frac{y_2(x)}{d}, \\
\bar{\lambda}(\bar{x}) &= \frac{\lambda(x)}{A_0}, \quad \bar{I}(\bar{x}) = \frac{I(x)}{I_0}, \quad \bar{t} = \frac{t}{T},
\end{align*}
\]
(2)

Substituting Eq. (2) into Eq. (1), the following non-dimensional equation of motion can be obtained:
\[
\frac{\partial^2}{\partial \bar{x}^2} \left( \bar{I}(\bar{x}) \frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}^2} \right) + \bar{\lambda}(\bar{x}) \frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{t}^2} + c \frac{\partial \bar{y}(\bar{x},\bar{t})}{\partial \bar{t}} - \alpha_2 \int_0^1 \bar{\lambda}(\bar{x}) \left( \frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}^2} \right)^2 dx \frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}^2} = 0
\]
(3)

In the following simplifications, the “\(^-\)” notation is dropped for convenience.

### 2.2 Galerkin Expansion

Since Eq. (3) is partial differential equation, only numerical solution can be obtained. The theoretical results can be obtained by using Galerkin method to discretize model into ordinary differential equations.

The deflection is expressed as:
\[
y(x,t) = \sum_{i=1}^n u_i(t) \phi_i(x). \tag{4}
\]

The boundary conditions are as follows:
\[
\phi_i(0) = \phi_i(1) = \phi_i'(0) = \phi_i'(1) = 0, \tag{5}
\]
where \( u_i(t) \) is the modal coordinate amplitude of the \( i \)-th mode. \( \phi_i(x) \) is the \( i \)-th mode shapes of the normalized undamped linear orthonormal. For an electrostatic actuated microbeam, a single degree-of-freedom model is sufficient to capture all the key nonlinear aspects in the Galerkin approximation [16]. The first-order modal vibration \( y(x,t) = u(t) \phi(x) \) is assumed. Substitute (4) into (3), upon multiplying by \( \phi(x) \) and integrating the outcome is from \( x = 0 \) to \( 1 \), one can obtain the following equation:
\[
\bar{u} + \mu \bar{u} + k_1 \bar{u} - \alpha_2 \int_0^1 \frac{\phi(x)}{(1-y_1(x)+\frac{h}{2d}-\phi(x)u)^2} dx - \alpha_2 \int_0^1 \frac{\phi(x)}{(1-y_1(x)+\frac{h}{2d}+\phi(x)u)^2} dx + 2\alpha_4 \rho \sin(\omega t) \int_0^1 \frac{\phi(x)}{(1-y_1(x)+\frac{h}{2d}-\phi(x)u)^2} dx + 2\alpha_4 \rho \sin^2(\omega t) \int_0^1 \frac{\phi(x)}{(1-y_1(x)+\frac{h}{2d}+\phi(x)u)^2} dx, \tag{6}
\]

where \( \bar{u} = du/dt \). The symbolic meanings of \( \mu \), \( k_1 \) and \( k_2 \) are discussed as follows:
g = \int_0^1 A(x) \phi'(x) dx, \quad \mu = c / g, \quad k_i = \int_0^1 (I(x) \phi'(x)) \phi(x) dx / g, \quad k_3 = \int_0^1 A(x)(\phi'(x))^2 dx \int_0^1 \phi'(x) \phi(x) dx / g.

The integral of the electrostatic force in Eq. (6) is complicated. Therefore, the Newton–Cotes method is applied to fit the electrostatic force [17]. The final mathematical model is
\[
\dot{u} + \mu \dot{u} + k_i u - \alpha_3 k_i u^3 = \alpha_1 \frac{0.61}{(1 - \delta^2 - 1.48u)} - \frac{0.61}{(1 - \delta^2 + 1.48u)^2},
\]
\[
+ 2\alpha_2 \rho \sin(\omega t) + \alpha_2 \rho^2 \sin^2(\omega t) + \alpha_3 \rho^3 \sin^2(\omega t) + \alpha_4 \rho^4 \sin^2(\omega t).
\]
where, \( \delta = h / d \).

The degree of matching is illustrated in Fig. 2. Transverse coordinate is displacement and the ordinate is integral term of electrostatic force. It can be seen that both results are accurate before the pull-in phenomenon. Therefore, the results obtained by the Newton-Cotes method are accurate.

![Figure 2: Contrast diagram of fitting curves under different section parameters. The circle is calculated using the numerical solution. The line is calculated using the Newton-Cotes method.](image)

The static equilibrium equation can be written:
\[
k_i u - \alpha_3 k_i u^3 = \alpha_1 \frac{0.61}{(1 - \delta^2 - 1.48u)} - \frac{0.61}{(1 - \delta^2 + 1.48u)^2}.
\]

It should be noted here that the maximum lateral displacement of the microbeam is at the midpoint. After the nondimensional obtained
\[
y_{max} = \phi(0.5)u \in [1 + \lambda \delta, 1 - \lambda \delta].
\]
The value of the modal function is \( \phi(0.5) = 1.59 \). Therefore, the range of \( u \) is \( u \in [1 - 1.59 \delta, 1 - 1.59 \delta] \).

3. Perturbation analysis

Since \( V_\infty \) is far less than \( V_c \) in the microresonator, the terms \( V_\infty = O(1) \) and \( V_c = O(\varepsilon^3) \) are considered. Here, \( \varepsilon \) is regarded as a small non-dimensional bookkeeping parameter. Therefore, Eq. (8) can be modified as:
\[
\dot{u} + \varepsilon^2 \mu \dot{u} + \alpha_3 u^3 + \alpha_4 u^2 + \alpha_5 u = \varepsilon f(\omega t).
\]

The symbolic meanings of \( \alpha_1, \alpha_2, \alpha_3 \) and \( f \) are discussed as follows:
\[
\alpha_1 = \left\{ k_1 - 3\alpha_2 k_3 x^3 - \alpha_3 \left( \frac{1.8056}{1 - 1.48x - h/\lambda} + \frac{1.8056}{1 + 1.48x - h/\lambda} \right) \right\}^{0.5},
\]
\[
\alpha_2 = -3\alpha_2 k_3 x - \alpha_3 \left( \frac{4.00843}{1 - 1.48x - h/\lambda} - \frac{4.00843}{1 + 1.48x - h/\lambda} \right) \}
\]
\[
\alpha_3 = -\alpha_3 k_3 - \alpha_3 \left( \frac{7.90997}{1 - 1.48x - h/\lambda} + \frac{7.90997}{1 + 1.48x - h/\lambda} \right),
\]
\[
f = 2 \times 0.61 \alpha_5 \rho \left( \frac{1}{1 - h/\lambda - 1.48x} \right).
\]
To describe the nearness of the primary resonance, a detuning parameter $\sigma$ is introduced and defined by

$$\omega = \omega_n + \varepsilon^2 \sigma. \quad (10)$$

The approximate solution of Eq. (9) can be obtained in the following form:

$$u_n(t, \varepsilon) = \omega_n u_{m1}(T_0, T_1) + \varepsilon^2 u_{m2}(T_0, T_1) + \varepsilon^3 u_{m3}(T_0, T_1), \quad (11)$$

where $T_n = \varepsilon^n t$, $n = (0, 1, 2)$.

Substitute Eqs. (10) and (11) into Eq. (9) and equating the coefficients of like powers of $\varepsilon$, the following equations can be obtained:

$$O(\varepsilon^3): D_n^3 u_{m1} + \omega_n^2 u_{m1} = 0, \quad (12)$$

$$O(\varepsilon^2): D_n^3 u_{m2} + \omega_n^2 u_{m2} = -2D_n D_n u_{m1} - a_n u_{m1}, \quad (13)$$

$$O(\varepsilon): D_n^3 u_{m3} + \omega_n^2 u_{m3} = -2D_n D_n u_{m2} - 2D_n D_n u_{m1} - D_n^3 u_{m1} - \mu D_n u_{m1} - 2a_n u_{m2} - a_n u_{m1} + f \cos(\omega_n T_n + \sigma T_n), \quad (14)$$

where $D_n = \frac{\partial}{\partial T_n}$, $n = (0, 1, 2)$.

The general solution of Eq. (12) can be written as:

$$u_{m1}(T_0, T_1, T_2) = A(T_1, T_2) e^{\omega_n T_n} + \tilde{A}(T_1, T_2) e^{-\omega_n T_n}. \quad (15)$$

Substituting Eq. (15) into Eq. (13), yields:

$$D_n^3 u_{m2} + \omega_n^2 u_{m2} = -2i\omega_n \frac{\partial A}{\partial T_1} e^{\omega_n T_n} - a_n (A^2 e^{2\omega_n T_n} + \tilde{A}) + cc, \quad (16)$$

where cc represents the complex conjugate terms.

To eliminate the secular term, one needs

$$-2i\omega_n \frac{\partial A}{\partial T_1} e^{\omega_n T_n} = 0, \quad (17)$$

which indicates that $A$ is only a function of $T_1$.

Thus, Eq. (16) becomes

$$D_n^3 u_{m2} + \omega_n^2 u_{m2} = -a_n (A^2 e^{2\omega_n T_n} + \tilde{A}) + cc. \quad (18)$$

The solution of $u_{m2}$ can be given as

$$u_{m2}(T_0, T_1) = \frac{a}{3\omega_n} A^2 e^{2\omega_n T_n} - \frac{a}{\omega_n} \tilde{A} + cc. \quad (19)$$

Substituting Eqs. (15) and (19) into Eq. (14) yields the secular terms

$$2i\omega_n \frac{\partial A}{\partial T_1} + \mu i\omega_n A - \frac{10a_n^2 A^2 \tilde{A}}{3\omega_n^3} + \frac{3a_n A^2 \tilde{A}}{\omega_n} - \frac{f}{2} e^{\omega_n T_n} = 0. \quad (20)$$

At this point, it is convenient to express $A$ in the polar form

$$A = \frac{1}{2} a (T_1) e^{i(\sigma T_1 + \beta)} + cc. \quad (21)$$

Substitute Eq. (21) into Eq. (20), and separating the imaginary and real parts, yield

$$\frac{DA}{DT_1} = \mu a + \frac{f}{2\omega_n} \sin \varphi, \quad (22)$$

$$\frac{D\varphi}{DT_1} = \sigma a + a \left( \frac{5a_n^2}{12\omega_n^3} \right) - \frac{3a_n}{8\omega_n} + \frac{f}{2\omega_n} \cos \varphi, \quad (23)$$

where $\varphi = \sigma T_1 - \beta$.

The steady-state response can be obtained by imposing the conditions: $\frac{DA}{DT_1} = \frac{D\varphi}{DT_1} = 0$. Finally, the frequency response equation can be derived as follows:
\[ a^2 \left( \frac{\mu}{2} \right)^2 + (\sigma + a^2 \kappa)^2 = \left( \frac{f}{2\omega_n} \right)^2, \]  

(24)

where \( \kappa = \frac{5a_2^2}{12a_1^2} - \frac{3a_1}{8a_n} \).

The stability of the periodic solution can be determined by the method in [18].

The frequency response is very important for researching the properties of the microbeam. The resonant frequency, amplitude, and softening and hardening behaviors can intuitively observed through frequency response. The following physical quantities are assumed: \( l = 400 \mu m \), \( b = 45 \mu m \), \( h = 2 \mu m \), \( d = 3 \mu m \), \( E = 165 \text{ GPa} \), \( \rho = 2.33 \times 10^3 \text{ kg/m}^3 \), \( \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m} \), \( \varepsilon_s = 1 \) and dimensionless damping coefficient \( \mu = 0.1 \). Then the effect of section parameter \( \lambda \) on frequency response is discussed. The analytical solution is obtained by MMS and the long-time integration method (LTI) of Eq. (7) is used to obtain numerical solutions. The accuracy of the results are verified by comparing both results. The relationship between section parameter and hardware and software behaviors are shown in Fig. 3-a. It can be seen that the changes from hardening to softening with the increase of \( \lambda \), that is, the increase of \( \lambda \) can promote the softening behavior. On the contrary, the decrease of \( \lambda \) can promote the hardening behavior. The frequency response in the three cases of \( \lambda = -0.1, \lambda = 0 \) and \( \lambda = 0.1 \) are plotted in Fig. 3-b. It can be seen that the frequency response is hardening behavior when \( \lambda = -0.1 \) and \( \lambda = 0 \), the frequency response curve is softening behavior when \( \lambda = 0.1 \). In addition, the resonance frequency decreases and the amplitude increases with the increase of \( \lambda \).

Figure 3: The influence of section parameter on nonlinear behaviors. (a) The relationship between section parameter and dimensionless parameter \( \kappa \). (b) Frequency response in three section parameters.

4. Parameter optimization

The influence of section parameter \( \lambda \) on frequency response is obtained in previous section. The results show that \( \lambda \) can change the softening and hardening behaviors of frequency response. It is noted that during the transformation of hardening and softening behaviors, there is a moment of linear vibration which is ideal condition for microresonator. Therefore, the microresonator is designed on this condition. The relationship of system physical parameters in linear vibration with different section parameters as shown in Fig. 4. The thickness of the clamped side has a negative correlation with the gap distance when the DC voltage is determined as shown in Fig. 4-a. The smaller the value of \( \lambda \) is, the more obvious the relationship becomes. Then the Fig. 4-b shows the relationship between clamped side thickness and DC voltage at \( d = 2.5 \mu m \). It can be known that the voltage must be increased to keep linear vibration with thickness increases. Because the structural stiffness increases with increasing thickness and the voltage increase causes the electrostatic force to increase. The softening effect of electrostatic force neutralizes structure stiffness into linear vibration. The relationship between clamped side clearance distance and DC voltage as shown in Fig. 6-c. The relationship has an opposite effect on the influence of clamped side thickness. The voltage must be decreased to keep...
linear vibration with gap distance increases.

Figure 4: The relationship of physical parameters in linear vibration with different sections parameters. (a) The relationship between the clamped side thickness and gap distance when \( V_{DC} = 50 \text{ V} \). (b) The relationship between the clamped side thickness and DC voltage when \( d = 2.5 \mu \text{m} \). (c) The relationship between the clamped side gap distance and DC voltage when \( h = 3 \mu \text{m} \).

In order to validate the optimization results, the parameters from Fig. 6-a are selected to simulate. The theoretical frequency obtained by Galerkin method are compared with the finite element method results obtained from COMSOL. The module used for this analysis is the MEMS module and the electrical physical field interface is selected. It can be seen from Table 1 that the theoretical results agree well with finite element results, and the discrepancy in the value of the resonance frequency between present method and the finite element method is less than 2\%, which demonstrates the present analytical and solving method are both effective.

<table>
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<th>Parameter ( \lambda )</th>
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<th>d (( \mu \text{m} ))</th>
<th>theoretical frequency(KHz)</th>
<th>Simulation frequency(KHz)</th>
<th>error</th>
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</table>

5. Conclusion

The doubly clamped microbeam-based resonator which actuated by two electrodes is investigated in this paper. The microresonator is optimized by adjusting the sectional shape. The mathematical model is simplified by Galerkin method and Newton-cotes method. Then the MMS is used to study the frequency response. It is found that the vibration changes from hardening to softening behaviors with the thickening of section. In the process of change will appear linear vibration which is ideal condition for microresonator. Based on this condition, the relationships between clamped side thickness, gap distance and DC voltage are obtained. The microresonator is optimized through the relationship. Finally, the finite element analysis is carried out using the software COMSOL for validating the accuracy of the theoretical results.

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