VIBRATION OF A BEAM ON AN ELASTIC FOUNDATION WITH A MOVING DISTRIBUTED MASS

Yun Liu
Faculty of Information and Automation, Kunming University of Science and Technology, Kunming 650051, China

Yin Zhang
*aState Key Laboratory of Nonlinear Mechanics (LNM), Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China.
bSchool of Engineering Science, University of Chinese Academy of Sciences, Beijing 100049, China
email: zhangyin@lnm.imech.ac.cn

Compared with the moving concentrated load model, it is more realistic and proper to use the moving distributed mass model to simulate the dynamics of a train moving along a railway track. In the problem of a moving concentrated load, there is only one critical moving velocity, which divides the load moving velocity into two categories: Subcritical and supercritical. The locus of a concentrated load demarcates the space into two parts: The waves in these two domains are called the front and rear waves, respectively. In comparison, in the moving distributed mass problem, there are two critical moving velocities, which results in three categories of the distributed mass moving velocity; the space is divided into three domains, in which three different waves exist. Much richer and different variation patterns of wave shapes arise in the moving distributed mass problem. The mechanisms responsible for these variation patterns are systematically studied.

Keywords: steady state, beam, elastic foundation, moving distributed mass

1. Introduction

The steady state solution of an infinite beam on an elastic foundation under a moving concentrated load was first obtained, in connection with the stress analysis of railway tracks, by Timoshenko in 1926 [1]. Timoshenko found that there is a critical moving velocity at which the vibration amplitude of the undamped beam approaches infinity [1]. With the (assumed) parameters of beam and foundation, this critical velocity is found to be around 1931 km per hour [1], which is much larger than the highest train speed at that time and nowadays. At a first look, the dynamic effect should be little because of this very high critical velocity [2]. While, this moving load induced vibration can cause very significant dynamic effect in conjunction with the track irregularities [3, 4, 5]. Recent analysis shows that the difference between the static and dynamic wheel-track contact stress can be as large as twenty times [4]. Nowadays the continuous welded rail (CWR) is widely used in modern railways. Because of the complete elimination of expansion joints in the CWR tracks, the thermal stresses due to the rising temperature can cause considerable axial compression, which leads to a significantly lower critical velocity or even buckling [2]. In fact, the possibility of the track buckling due to compressive thermal stress was the main reason for delaying
the use of CWR track by decades [2]. Nowadays, a large number of the train accidents are due to the track buckling instability [6]. Besides the compressive axial load, the soft soil, which is with small foundation modulus, can further reduce the critical velocity. It is not unusual that modern high-speed train exceeds this critical velocity [7]. Even as early as 1950s, this critical velocity was easily surpassed by the rocket-propelled vehicles on a test track [8]. The dynamic effects due to the rapidly moving load/mass become more and more important for the developments of high-speed train.

Physically, the moving concentrated load is to model the wheel-track contact load of a moving vehicle. Most studies adopted the model of a beam on an elastic foundation under one moving concentrated load [1, 2, 7, 8, 9, 10]. Clearly, with the moving load modeling, the mass/inertia effect of a moving vehicle is not embodied. Therefore, the model of a moving concentrated mass is proposed [11, 12, 13]. While, all these three models of an infinite beam show that the magnitude of a concentrated mass has no impact on the critical moving velocity [11, 12, 13], which is determined by the beam stiffness, foundation modulus and axial load only [2, 12, 13].

In this study, the model of an infinite beam on a viscoelastic foundation under a moving distributed mass is proposed to study the steady-state of a moving train. Because of the mass effect, there is one more critical moving velocity lower than the one as given by the moving concentrated load model. These two velocities, which are referred to as the critical upper and lower velocities, demarcate three velocity zones for a moving distributed mass: Subcritical, transcritical and supercritical regions, in which the qualitative differences of wave shapes arise. For a distributed mass moving along an infinite beam, besides the mass effect, it also cause the effect of a moving distributed load due to gravity. The use of the ballastless railway significantly improves the foundation modulus and thus a much higher critical upper velocity is also achieved. As shown later, this critical upper velocity can be easily higher than 2000 km per hour. While, with the consideration of the effect of a distributed mass, the critical lower velocity can be smaller than the operating speed of nowadays high-speed train, i.e., the train speed is in the transcritical region. This model of moving distributed mass shows a significantly different steady-state from that of a moving concentrated load, which can be of some help to a more accurate evaluation of the track vibration and stress.

2. Model development

In conjunction with the moving mass/load models in references [8,14], the corresponding governing equation for the beam vibration is given directly as the following:

$$EI \frac{\partial^4 y_1}{\partial x^4} + N \frac{\partial^2 y_1}{\partial x^2} + \left\{ m_1 + m_2 \left[ H \left( x_1 - vt + \frac{1}{2} \right) - H \left( x_1 - vt - \frac{1}{2} \right) \right] \right\} \frac{\partial^2 y_1}{\partial t^2} + c \frac{\partial y_1}{\partial t} + ky_1 = m_2 g \left[ H \left( x_1 - vt + \frac{1}{2} \right) - H \left( x_1 - vt - \frac{1}{2} \right) \right].$$

(1)

Where $y_1$, $E$, $I$ and $m_1$ are the beam deflection, Young’s modulus, area moment of inertia and mass per unit length, respectively. The beam is under an axial load of $N$ and here a positive/negative $N$ means compression/tension. The Winkler foundation is with the damping of $c$ and modulus of $k$. The distributed mass is with the mass per unit length of $m_2$, length of $l$ and a constant moving velocity of $v$. Here $H$ is the Heaviside unit function. The right-hand side term of Eq. (1), $m_2 g \left[ H \left( x_1 - vt + \frac{1}{2} \right) - H \left( x_1 - vt - \frac{1}{2} \right) \right]$, is the transverse distributed load exerted by the gravity of the distributed mass. As the distributed mass is moving along a vibrating path, the governing equation in the moving coordinate ($x - y$) is obtained as the following:

$$EI \frac{\partial^4 y}{\partial x^4} + N \frac{\partial^2 y}{\partial x^2} + \left\{ m_1 + m_2 \left[ H \left( x + \frac{1}{2} \right) - H \left( x - \frac{1}{2} \right) \right] \right\} \left( \frac{\partial^2 y}{\partial x^2} - 2v \frac{\partial^2 y}{\partial x \partial t} + v^2 \frac{\partial^2 y}{\partial x^2} \right) + c \frac{\partial y}{\partial t} - cv \frac{\partial y}{\partial x} +$$
\[ ky = m_2 g \left[ H \left( x + \frac{1}{2} \right) - H \left( x - \frac{1}{2} \right) \right]. \]  

(2)

For steady-state, all the time-related terms in Eq. (2) are gone, which leads to the following equation:

\[ EI \frac{\partial^2 y}{\partial x^2} + \left\{ N + m_1 v^2 + m_2 v^2 \left[ H \left( x + \frac{1}{2} \right) - H \left( x - \frac{1}{2} \right) \right] \right\} \frac{\partial^2 y}{\partial x^2} - cv \frac{\partial y}{\partial x} + ky = m_2 g \left[ H \left( x + \frac{1}{2} \right) - Hx - l \right]. \]  

(3)

In order to nondimensionalize Eq. (3), the following dimensional and dimensionless quantities are introduced:

\[ \lambda = \frac{k}{4EI}, \quad N_{cr} = 2\sqrt{kEI}, \quad v_{cr1} = \sqrt{\frac{4kEI}{m_1^2}}, \quad v_{cr2} = \sqrt{\frac{4kEI}{(m_1 + m_2)^2}}, \quad c_r = 2\sqrt{k/m_1}; \]

\[ Y = \lambda y, \quad \xi = \lambda x, \quad L = \lambda l, \quad \phi = \frac{N}{N_{cr}}, \quad \theta = \frac{v}{v_{cr1}}, \quad \beta = \frac{c}{c_r}, \quad \alpha = \frac{m_2}{m_1}, \quad \gamma = \frac{m_2 g \lambda}{k} = \frac{am_2 g \lambda}{k}. \]  

(4)

Figure 1: Schematic diagram of a uniformly distributed mass moving along a beam supported by a viscoelastic foundation.

Because of the finite length of the distributed mass, the governing equation of Eq. (3) is divided into two different equations in three zones as indicated by the Heaviside function. Eq. (3) is now nondimensionalized as follows

\[ \begin{align*}
EI \frac{\partial^2 y}{\partial \xi^2} + 4(\phi + \theta^2) \frac{\partial^2 y}{\partial \xi^2} - 80\beta \frac{\partial y}{\partial \xi} + 4Y &= 0, \quad \xi < -L/2 \quad \text{or} \quad \xi < -L/2, \\
EI \frac{\partial^2 y}{\partial \xi^2} + 4\left[ \phi + (1 + \alpha)\theta^2 \right] \frac{\partial^2 y}{\partial \xi^2} - 80\beta \frac{\partial y}{\partial \xi} + 4Y &= 0, \quad -L/2 \ll \xi \ll L/2.
\end{align*} \]  

(5)

The solution to the above equation set is the following, which is divided into three parts:

\[ \begin{align*}
Y_1(\xi) &= e^{-a\xi} \left[ C_1 \sin(b_2 \xi) + C_2 \cos(b_2 \xi) \right], \quad \xi > \frac{L}{2}, \\
Y_2(\xi) &= e^{-a\xi} \left[ C_3 \sin(b_1 \xi) + C_4 \cos(b_1 \xi) \right], \quad \xi < -\frac{L}{2}, \\
Y_3(\xi) &= e^{-c\xi} \left[ C_5 \sin(d_2 \xi) + C_6 \cos(d_2 \xi) \right] + e^{c\xi} \left[ C_7 \sin(d_1 \xi) + C_8 \cos(d_1 \xi) \right] + \gamma, \quad -\frac{L}{2} \ll \xi \ll \frac{L}{2}.
\end{align*} \]  

(6)

The eight unknowns of \( C_i \)s (i = 1 to 8) are determined by the eight boundary conditions at \( \xi = \pm \frac{L}{2} \), the continuity of the displacement, slope, moments and shear gives. And the exponents \( a \) and \( c \) are obtained as the positive real roots of the following two equations:

\[ \begin{align*}
a^6 + 2(\phi + \theta^2)a^4 + \left( (\phi + \theta^2)^2 - 1 \right)a^2 - \theta^2\beta^2 &= 0, \\
c^6 + 2(\phi + (1 + \alpha)\theta^2)c^4 + \left( (\phi + (1 + \alpha)\theta^2)^2 - 1 \right)c - \theta^2\beta^2 &= 0
\end{align*} \]  

(7)

(8)
The wave numbers of $b_1$, $b_2$ and $d_1$, $d_2$ are given as follows:

$$
\begin{align*}
    b_1 &= \sqrt{2(\phi + \theta^2) + a^2 - \frac{20\beta}{a}}, \\
    b_2 &= \sqrt{2(\phi + \theta^2) + a^2 + \frac{20\beta}{a}}, \\
    d_1 &= \sqrt{2[\phi + (1 + \alpha)\theta^2] + c^2 - \frac{20\beta}{c}}, \\
    d_2 &= \sqrt{2[\phi + (1 + \alpha)\theta^2] + c^2 + \frac{20\beta}{c}}.
\end{align*}
$$

(9)

3. Results and discussions

The following parameters are taken for a high-speed railway track (UIC60): $E = 2 \times 10^{11}$ N/m$^2$, $I = 3.06 \times 10^{-5}$ m$^4$ and $m_1 = 60.34$ kg/m. The foundation modulus varies in a large range of $5 \times 10^6$ N/m$^2 \leq k \leq 10^9$ N/m$^2$ and here a typical value of $k = 1.67 \times 10^7$ N/m$^2$ is taken. The corresponding quantities defined in Eq. (4) are with the following fixed values: $\lambda = 0.8991$ m$^{-1}$, $N_{cr} = 1.979 \times 10^7$ N, $v_{cr1} = 572.7$ m/s (2061.7 km per hour) and $c_r = 62134$ kmg$^{-1}$s$^{-1}$.

We firstly compute the case $\alpha = 50$ and $L = 2 \times 10^{-3}$. Here the very small $L$ is taken to simulate the concentrated load and mass scenario. The wave shapes of $\theta = 0$, $\theta = 1$ and $\theta = 2$ with the damping fixed as $\beta = 0.1$ are plotted in Fig. 2. The wave shapes of $\theta = 1$ and $\theta = 2$ for a moving concentrated load problem are also computed by Kenney [8]. It is noticed that in Fig. 2, the beam static deflection is symmetric and other two deflections (or wave shapes) are asymmetric. For both $\theta = 1$ and $\theta = 2$, the wave numbers of the front waves ($\xi > L/2 \approx 0$) are larger than those of the rear waves ($\xi < -L/2 \approx 0$, or say, the wavelengths of the front waves are smaller than those of the rear waves. As seen in Eq. (9), the wave numbers of the front and rear waves are $b_2$ and $b_1$, respectively. The difference between $b_2$ and $b_1$ is their last term of $20\beta/a$. Because $\theta$ and $\beta$ are non-negative, $b_2 \geq b_1$. As shown later, the exponent of a decreases monotonically as the moving velocity $\theta$ increases. As a result, the difference between $b_2$ and $b_1$ enlarges as $\theta$ increases, which then leads to a larger asymmetry in their wave shapes. While, it is also noticed that $\theta = 1$ is with the largest amplitude. The reason is that $\theta = 1$ is the critical velocity and thus the resonance point of the undamped system of a beam on an elastic foundation [8]. The resonance point of the damped system with $\beta = 0.1$ slightly shifts to a $\theta$ value smaller than 1. Compared with $\theta = 0$ and $\theta = 2$, $\theta = 1$ is closest to the resonance point and therefore, it has the largest amplitude. Besides the moving velocity $\theta$, damping also plays an important role of determining the asymmetry. Chen and Chen concluded that damping is the (only) mechanism responsible for the symmetry-breaking [29]. When $\beta = 0$ and $\theta < 1$, $b_2 = b_1$ and the symmetry is thus reserved no matter what the moving speed is. While, $\theta = 0$ can also lead to $b_2 = b_1$. Furthermore, the Kenney’s approximate analytical solution shows that even with $\beta = 0$, $b_2$ and $b_1$ will bifurcate into two different values when $\theta$ reaches the critical velocity and becomes supercritical [8]. Mathematically, the asymmetry is determined by the $20\beta/a$ term and $a$ is a function of $\theta$ and $\beta$. It is safe to conclude that both the moving velocity and damping determine the asymmetry. As shown in Fig. 2, one characteristics of the asymmetry is that most of the beam deflection occurs behind the load. Physically, this is caused by that the energy of the deformation in the beam/foundation system propagates with an average velocity (considering all waves with different wavelengths) lower than that of the moving load [7]. It is also noticed that due to the presence of damping, the maximum beam displacement is no longer at the point of $\xi = 0$, but a point behind. As $L$ approaches zero, the distributed mass model becomes a concentrated one. Although the effect of distributed mass is mathematically indicated by $\alpha$ in Eq. (5), our above results recover those of a moving concentrated load [8] as if only the load effect ($\gamma$) is embodied. Therefore, it confirms again that the concentrated mass has no impact on the critical velocity [11, 12, 13]. Furthermore, Fryba proved that for the steady-state of an infinite beam traversed by a moving concentrated mass, the mass exerts no inertia effects [14].
Figures 3 plots the beam wave shapes at $\theta = 0.8$ for $\alpha = 5$. This $\theta = 0.8$ is a supercritical velocity in the distributed mass area but a subcritical one in other areas and here we call it a transcritical one. As seen in Fig. 3(a), the wave shapes are (almost) symmetric to $\xi = 0$ and with one dominant wave number ($d_1$) for very small damping of $\beta = 10^{-3}$. As $\beta$ changes from $10^{-3}$ to 0.8, the wave numbers are slightly changed. With the presence of larger damping of $\beta = 0.2$ and 0.8, the waves become asymmetric: the waves with the wave number of $d_2$ cluster on the left and the waves with $d_1$ cluster on the right. As seen in Fig. 3, the wave amplitudes monotonically decrease and the asymmetry becomes more outstanding with the increase of damping. Compared with the asymmetry pattern in Fig. 2, there is a fundamental difference in Fig. 3. As mentioned above, in Fig. 2 shorter wave is ahead of concentrated load and the longer wave is behind. While, things are reversed in the distributed mass area of $-L/2 \leq \xi \leq L/2$ as seen in Fig. 3: shorter wave in now behind in the area of $-L/2 \leq \xi \leq 0$ and longer wave is ahead in the area of $0 \leq \xi \leq L/2$.

Figure 3: The wave shapes with $\alpha = 5$ at $\theta = 0.8$ with different dampings.
Figure 4 plots the beam wave shapes at $\theta = 1.2$ for $\alpha = 5$ and $\alpha = 10$, respectively. This $\theta = 1.2$ value exceeds 1, which is the supercritical case. In Fig. 4, the symmetric and asymmetric patterns inside the distributed mass area of $-L/2 \leq \xi \leq L/2$ follow the same trends as those in Fig. 3. The main pattern differences between the waves with $\theta > 1$ (supercritical) and the waves with $\theta = 0.8$ (transcritical) are in the areas of $\xi < -L/2$ and $\xi > L/2$, especially when $\beta$ is small. These differences are most outstanding in Figs. 4(a): The wave with a large wave number of $b_2$ (or a small wavelength) arises in the $\xi > L/2$ area; the wave with a small wave number of $b_1$ (or a large wavelength) arises in the $\xi < -L/2$ area. Their wave amplitudes are also significant. These difference can be explained by the solution forms of wave shape as given by Eq. (6).

4. Conclusion

In the moving distributed mass problem, there are two critical moving velocities and the space is divided into three domains. In these three domains, there are three different waves with four different wave numbers ($b_1$, $b_2$, $d_1$, and $d_2$) and two different exponents ($a$ and $c$), which determine the wave shapes. The two critical moving velocities determine the variation patterns of the wave numbers and exponents: The wave numbers bifurcate and the exponents rapidly approach zero at each critical velocity for the no damping case. Larger damping results in the large exponents of both $a$ and $c$, which leads to the larger wave amplitude reduction both inside and outside the distributed mass area; larger damping makes some wave numbers ($b_2$ and $d_2$) larger and some ($b_1$ and $d_1$) smaller. As a result, these variation patterns of the wave numbers and exponents cause significant wave shape changes as the moving velocity varies. The moving velocity and damping play a major role in determining the symmetry/asymmetry of the wave shapes. Compared with the moving concentrated load problem, there is no “resonance” of the wave amplitude response to the moving velocity in the moving distributed mass problem.
REFERENCES


