THE SECOND SPECTRUM IN TIMOSHENKO BEAM THEORY: A NEW APPROACH FOR ITS IDENTIFICATION

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The problem of existence and identification of a second frequency spectrum in the Timoshenko beam theory is reconsidered from a novel perspective, studying a Timoshenko beam rotating with constant angular speed about its longitudinal axis. The existence of a second spectrum in the case of non–rotating beams and general boundary conditions has been much debated in the literature, since it is possible to easily identify the companion natural frequencies constituting the second spectrum only in particular cases. Its existence in a non–rotating finite–length beam has been recently demonstrated on the basis of accurate experimental results, at least for free–free boundary conditions, and also by considering free waves in beams of infinite length. In this study new evidence of existence of a second spectrum together with a novel definition for its identification are presented, possible when considering gyroscopic effects. As a secondary result, it can be stated that the whole second spectrum gives no contribution to the forward critical speeds of the rotating beam.

Keywords: Timoshenko beam theory, second spectrum.

1. Introduction

The existence of two distinct spectra of frequencies for the free flexural vibrations of beams described by Timoshenko beam theory [1] was first claimed in 1953 [2], at least for some particular boundary conditions in which the characteristic equation factorizes, yielding two distinct natural frequencies associated to the same modal shape.

Since then, the existence of a second frequency spectrum in the case of general boundary conditions has been much debated in the literature [3, 4], mainly due to the difficulty in classifying the natural frequencies of the two distinct spectra when the characteristic equation does not factorize. In fact, attempts made by means of finite element simulations produced conflicting conclusions [5].

According to some authors, the whole second spectrum should be disregarded, and considered un–physical [4, 5]. However, more recently the existence of a second frequency spectrum in a non–rotating finite–length beam has been demonstrated on the basis of accurate experimental results, at least for free–free boundary conditions [6], and also by considering free waves in beams of infinite length, showing the existence of two distinct frequency branches for any wavenumber [7].

In this study a different proof for the existence of a second frequency spectrum, together with a novel definition for its identification are presented, possible in presence of gyroscopic effects. The general case of a Timoshenko beam rotating with constant angular speed about its longitudinal axis and loaded by constant axial end thrust and twisting moment is considered [8], investigating also the role of the so–called cut–off Timoshenko beam frequencies [4], and extending their definition to include the effects of gyroscopic moments and external loads.

2. Model description and equations of motion

A homogeneous uniform Timoshenko straight beam with circular section is considered, rotating at constant angular speed about its longitudinal axis and simultaneously subjected to axial end thrust and twisting moment.
The model is characterized by the following parameters:

\[ A = \pi r^2 \text{ cross-sectional area [m}^2] \]
\[ l = \text{length of the shaft [m]} \]
\[ E = \text{Young's modulus [N/m}^2] \]
\[ G = \text{shear elasticity modulus [N/m}^2] \]
\[ I_x = I_y = J = \text{moment of inertia of the cross-section [m}^4] \]
\[ I_z = 2J = \text{polar moment of inertia of the cross-section [m}^4] \]
\[ \kappa = \text{transverse shear factor} \]
\[ \nu = \text{Poisson's ratio} \]
\[ \rho = \text{density [Kg/m}^3] \]
\[ \omega = \text{rotating angular speed [rad/s]} \]

The external loads \( N \) (positive if tensile) and \( T \) (positive if counterclockwise) are assumed constant with respect to time. Isotropic supports are considered, making the whole model axisymmetric. Hence it can be represented in a non-rotating coordinate system as shown in Fig. 1.

![Figure 1: Schematic representation of the model.](image)

Additional nomenclature includes:

- \( u, v, w \) = displacements in the \( x, y, z \) directions [m]
- \( \bar{w} \) = complex displacement [m]
- \( \vartheta_x, \vartheta_y, \vartheta_z \) = angular displacements about the \( x, y, z \) axes [rad]
- \( \theta = \vartheta_y + i\vartheta_z \) = complex angular displacement [rad]

Adopting a notation with dots and roman numbers for differentiation with respect to time and to the spatial variable \( x \), then the equations of motion for the flexural dynamic behaviour of the shaft take the form [8]:

\[
\begin{align*}
\rho A \ddot{w} - \kappa G A (\dot{v}^2 - \dot{w}^2) - N \dot{w}^2 &= 0 \\
\rho A \ddot{\bar{w}} - \kappa G A (\dot{\bar{w}}^2 + \dot{\bar{w}}^2) + N \dot{\bar{w}}^2 &= 0 \\
\rho J \ddot{\vartheta}_x + 2\rho J \omega \dot{\vartheta}_x + (\kappa G A - N)(\dot{\vartheta}_x + \dot{\vartheta}_z) - EJ \ddot{\vartheta}_x - T \dot{\vartheta}_z &= 0 \\
\rho J \ddot{\vartheta}_z - 2\rho J \omega \dot{\vartheta}_z - (\kappa G A - N)(\dot{\vartheta}_x - \dot{\vartheta}_z) - EJ \ddot{\vartheta}_z + T \dot{\vartheta}_x &= 0
\end{align*}
\]

Introducing the complex displacements \( w \) and \( \theta \), the four equilibrium equations Eqs. (1) can be decoupled into two fourth-order partial derivative equations with complex coefficients. A dimensionless spatial variable \( \xi \) and a dimensionless time \( \tau \) are considered, along with a reference frequency \( \Omega \) (representing the structural properties of the shaft) plus five dimensionless parameters, defined as:

\[
\xi = \frac{x}{l}, \quad \tau = \Omega t, \quad \Omega = \frac{1}{l} \sqrt{\frac{EJ}{\rho A}}, \quad \alpha = l \sqrt{\frac{A}{l} \frac{2I}{r}}, \quad \sigma = \frac{E}{\kappa G}, \quad \omega = \frac{\omega}{\Omega}, \quad \dot{N} = \frac{N}{EA}, \quad \dot{T} = \frac{Tl}{EJ}
\]

where \( \alpha \) is the slenderness ratio. Then the fourth-order partial derivative equation for the complex displacements \( w \) can be written in the following non-dimensional form [8]:

\[
\left( \frac{w''' - \sigma}{\alpha^2} \right) - i\dot{T} \left( \frac{w'' - \sigma}{\alpha} \right) - \dot{\bar{N}} \alpha^2 \psi w'' + \psi w - \frac{1}{\alpha^2} \left( \frac{w'' - \sigma}{\alpha} \right) + \frac{2I}{\alpha^2} \ddot{\vartheta}_z \left( \frac{w'' - \sigma}{\alpha} \right) = 0, \quad \psi = 1 - \sigma \dot{N}
\]

\[
(3) \quad \left( \frac{w''' - \sigma}{\alpha^2} \right) - i\dot{T} \left( \frac{w'' - \sigma}{\alpha} \right) - \dot{\bar{N}} \alpha^2 \psi w'' + \psi w - \frac{1}{\alpha^2} \left( \frac{w'' - \sigma}{\alpha} \right) + \frac{2I}{\alpha^2} \ddot{\vartheta}_z \left( \frac{w'' - \sigma}{\alpha} \right) = 0, \quad \psi = 1 - \sigma \dot{N}
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\]
Separating the variables and Laplace transforming with respect to time yields:

\[ w(\xi, \tau) = \phi(\xi)\eta(\tau) \Rightarrow L(w) = \phi(\xi)\eta(s) \Rightarrow p_4\phi^{IV} + p_3\phi^{III} + p_2\phi^{IV} + p_1\phi + p_0\phi = 0 \]

with \( p_4 = 1 \), \( p_1 = -i\hat{T} \), \( p_2 = \left[ \frac{1 + \sigma}{\alpha} s^2 - 2i\hat{\omega}\hat{\psi} + \hat{N}\alpha^2\psi \right] \), \( p_3 = i\hat{T} \frac{\sigma}{\alpha} s^2 \), \( p_0 = s^{2}\left( \frac{\sigma}{\alpha} s^2 - 2i\hat{\omega}\hat{\psi} + \hat{N}\alpha^2\psi \right) \)

(4)

where \( p_2, p_1 \), and \( p_0 \) depend on the (dimensionless) eigenvalue \( s \). Solutions can be expressed on the basis of the exponential function, and since the characteristic equation for the exponents is a quartic polynomial with complex coefficients \( p \), the general integral is given by a linear combination of four complex exponential functions. Hence the eigenfunctions for the complex displacements \( w \) and \( \theta \) take the form:

\[ \phi_n(\xi) = \sum_{n=1}^{4} B_n e^{\alpha_n \xi}, \quad \psi_n(\xi) = \sum_{n=1}^{4} B_n R_n(\lambda) e^{\alpha_n \xi}, \quad R_n(\lambda) = \frac{1}{\psi a_n} \left( a_n^2 + \frac{\sigma \lambda^2}{\alpha} \right) \]

(5)

After setting four boundary conditions for coefficients \( B_n \), pure imaginary eigenvalues \( s = i\hat{\lambda} \) can be numerically computed by using a zero–find routine of a real function \( f \) in the real variable \( \lambda \):

\[ f[\Delta(i\hat{\lambda})] = 0, \quad \lambda \in (-\infty, +\infty) \]

(6)

The critical speeds can be found following the same procedure, setting \( \hat{\lambda} = \hat{\omega} \) in Eq. (6).

3. Considerations on vibration modes

A qualitative analysis of the four exponents \( a_n \) in Eq. (5) highlights some general aspects of modal shapes, independently from boundary conditions. The real and imaginary parts of the four roots \( a_n \) can be displayed in 3D plots as functions of a continuous variable \( \lambda \), representing all possible natural frequencies, as for instance reported in Fig. 2 for \( \alpha = 10 \), \( \sigma = 2.933 \) (due to a Poisson’s ratio \( \nu = 0.3 \) in the case of a homogeneous beam with circular section made of isotropic material [8]), \( \hat{\omega} = 50 \), \( \hat{N} = 0.005 \) and \( \hat{T} = 0 \). If \( \hat{T} = 0 \), the characteristic equation for the exponents \( a_n \) is a biquadratic equation, with either two real opposite and two imaginary conjugate roots, or two pairs of imaginary conjugate roots. The two non–zero frequency values for which two pairs of real roots \( a_n \) become null (and then switch to imaginary conjugate, in the following referred to as switch frequencies) can be found by setting \( p_0 = 0 \) in Eq. (4), yielding:

\[ p(\lambda) = \frac{\sigma}{\alpha} \lambda^2 - 2 \frac{\sigma \hat{\omega}}{\alpha^2} \lambda - \psi = 0 \Rightarrow \begin{cases} \lambda_+ = \hat{\omega} + \sqrt{\hat{\omega}^2 + \psi \alpha^2 \sigma^2} \\ \lambda_- = \hat{\omega} - \sqrt{\hat{\omega}^2 + \psi \alpha^2 \sigma^2} \end{cases} \]

(7)

Figure 2. The four exponents \( a_n \) of the modal shapes, as functions of natural frequencies \( \lambda \).
Inside the interval defined by the two switch frequencies (\(\lambda_b, \lambda_f\)), the modal shapes can be defined by combinations of hyperbolic and trigonometric functions; outside this range, the modal shapes are represented by trigonometric functions only. At the switch frequencies the eigenfunctions take a peculiar form: \(\phi_w = 0\) (the elastic line does not bend) and \(\phi_0 = \text{constant}\) (constant angular displacements along the spatial coordinate \(\xi\)).

If a twisting moment acting on the shaft is considered (\(\hat{T} \neq 0\)), then the only preserved plane of symmetry is the \(\lambda–\text{Im}(a)\) plane. Symmetry with respect to the \(\lambda–\text{Re}(a)\) plane is lost, as shown in Fig. 3 for a rotating shaft with \(\alpha = 10, \sigma = 2.933, \omega = 50, N = 0\) and several increasing values of \(\hat{T} > 0\). In Fig. 3 (left), to emphasize the effects of \(\hat{T}\) on the overall behaviour of the roots \(a_n\) in the \(\lambda–\text{Im}(a)\) plane, the twisting moment is increased up to exceedingly high values, the represented map retaining mathematical meaning only. In Fig. 3 (right), a small portion of the same plot is displayed, around the forward switch frequency values, varying \(\hat{T}\) in a realistic range. The points in which the curves cross the \(\lambda–\text{axis}\) are independent from \(\hat{T}\), and it can be demonstrated that their frequency values are still given by Eqs. (7). In these points (and not at the actual switch frequencies for \(\hat{T} \neq 0\)), the modal shapes retain the already described features (\(\phi_w = 0\) and \(\phi_0 = \text{constant}\)).

Regarding the behaviour at high frequency, it turns out that in the most general case (\(\hat{\omega} \neq 0, \hat{N} \neq 0, \hat{T} \neq 0\)) the pairs of conjugate imaginary roots \(a_n\) are asymptotic to straight lines, given by:

\[
a = \pm \frac{i \sqrt{\alpha}}{\alpha} \lambda \quad \text{always pure imaginary roots}
\]

\[
a = \pm i \frac{\lambda}{\alpha} \quad \text{pure imaginary roots beyond the switch frequencies}
\]

which depend strongly on slenderness ratio \(\alpha\), but they are independent from \(\hat{\omega}, \hat{N}\) and \(\hat{T}\). Therefore the effects of external loads are the largest on lower modes, while progressively fading away at increasing frequencies.

Figure 3. The imaginary parts of the four exponents \(a_n\) of the modal shapes, as functions of the natural frequencies \(\lambda\); left: overall diagram with several increasing values of \(\hat{T} > 0\) (\(\hat{\omega} > 0, \hat{N} = 0\)); right: detail of the position of switch points for realistic values of \(\hat{T} > 0\).

4. Second spectrum identification

The second frequency spectrum in the Timoshenko beam theory can be identified by considering the asymptotic behaviour of the eigenvalues with respect to angular speed, in a general case. Divi-
ding the third and fourth of Eqs. (1) by \( \omega \) and letting \( \omega \to \infty \), then in these two equations all terms but the gyroscopic ones vanish. Consequently, if eigenvalues tend to finite values, there are two possible cases: either \( \lambda \to \lambda_\omega \), \( \lambda_\omega \neq 0 \), in which limit case Eqs. (1) yield:

\[
\begin{align*}
\omega \to \infty & \quad \Rightarrow \begin{cases}
\vartheta_y = 0 \\
\vartheta_z = 0
\end{cases} \quad \Rightarrow \begin{cases}
\rho A \ddot{v} - \kappa GA v^3 = 0 \\
\rho A \ddot{w} - \kappa GA w^3 = 0
\end{cases} \quad \Rightarrow \begin{cases}
\dot{\lambda}_\omega = \frac{\alpha}{\sqrt{\sigma}} a_{n,h}, \\
a_{n,h} = \sqrt{-\frac{\alpha}{\sigma}} \langle \phi_{n,h}, \psi \rangle
\end{cases}
\end{align*}
\]  

(9)

(where \( h \) identifies the mode order and \( \langle \cdot, \cdot \rangle \) an inner product involving the real variable eigenfunctions \( \phi \) related to the real variables \( v, w \)) or \( \lambda \to 0 \), in which limit case Eqs. (1) reduce to:

\[
\begin{align*}
\omega \to \infty & \quad \Rightarrow \begin{cases}
v_y - \psi \vartheta_y = 0 \\
w_y + \psi \vartheta_y = 0
\end{cases}
\end{align*}
\]  

(10)

If, on the other hand, eigenvalues tend to infinity with angular speed, dividing the third and fourth of Eqs. (1) by \( \omega^2 \) and letting both \( \omega \to \infty \) and \( \lambda \to \infty \) gives:

\[
\begin{align*}
\omega \to \infty \quad \Rightarrow \quad & \lambda = k_{e,\omega} \quad \text{with} \\
\lambda \to \infty & \quad \Rightarrow \begin{cases}
k_e = 2 & \text{if} \quad I_y = 2J, \quad I_z = I_x = J \\
k_e = I_y / J & \text{if} \quad I_y = I_x = J
\end{cases}
\end{align*}
\]  

(11)

Therefore the asymptotic behaviour of the forward and backward natural frequencies of the rotating shaft can be summarized as:

\[
\begin{align*}
\omega \to \infty & \quad \Rightarrow \begin{cases}
\dot{\lambda} = 0, & \quad \text{horizontal asymptote} \\
|\dot{\lambda}| = \frac{\alpha}{\sqrt{\sigma}} a_{n,h}, & \quad \text{horizontal asymptote} \\
\lambda \to \infty, & \quad \text{asymptotic to} \quad \lambda = k_{e,\omega}
\end{cases}
\end{align*}
\]  

(12)

The horizontal asymptotes in Eqs. (12) correspond to the straight asymptotic lines defined in the first of Eqs. (8), which in the case of finite eigenvalues are reached as \( \omega \to \infty \).

![Figure 4. Absolute values \( |\dot{\lambda}| \) (black continuous lines) of the two pairs of natural frequencies related to \( h = 1 \) (left) and \( h = 2 \) (right) as functions of \( \dot{\omega} \) (\( \alpha = 20, \sigma = 2.933, \dot{N} = \dot{T} = 0 \), simply supported ends). Grey continuous lines indentify asymptotes, grey dotted curves identify the switch frequencies.](image-url)
Figure 4 shows the absolute values $|\lambda|$ (black continuous lines) of the two pairs of eigenvalues related to $h=1$ (left) and $h=2$ (right) as functions of $\dot{\omega}$, with $\alpha=10, \sigma=2.933 (\nu=0.3), \dot{\omega}=50, \dot{N}=\dot{T}=0$ and simply supported ends. Grey continuous lines indentify asymptotes; intersections with the grey dotted line identify critical speeds on the Campbell diagram.

Notice that the asymptotic behaviour does not depend on the external loads $N$ and $T$, and that for a shaft with same boundary conditions at both ends, from Eq. (9) it results simply:

$$a_{\infty} = h\pi$$

as in the case of Fig. 4.

Each non–zero horizontal asymptote represents a link between two pairs of eigenvalues, one pair at lower frequencies, one pair at higher frequencies. The latter, when $\omega=0$ and in particular cases of boundary conditions in which the characteristic equation factorize (as in the case of simply supported ends), can be identified with the Timoshenko second spectrum.

However, when considering a rotating shaft, as $\omega \to \infty$ the existence of non–zero horizontal asymptotes for any boundary conditions suggests a new way for defining and identifying the natural frequencies of the Timoshenko first and second spectra.

All first spectrum backward eigenfrequencies tend to 0; all second spectrum forward eigenfrequencies tend to infinity, asymptotic to $\lambda=k\omega$, while the absolute value of each first spectrum forward eigenfrequency converges to the backward companion one belonging to the second spectrum.

Therefore, the first spectrum can be identified by setting $\lambda=k\omega, 0<k\leq k_s$ in Eq. (6) and solving it with respect to $\dot{\omega}$. The solutions identify the curves, or branches, of the first spectrum forward eigenvalues, since those of the second spectrum do not intersect any of the lines $\lambda=k\omega, 0<k\leq k_s$, as for example shown in Fig. 5. As a consequence, notice also that the whole second spectrum gives no contribution to the forward critical speeds of the shaft.

The problem of identifying the first spectrum frequencies at a given angular speed (eventually at $\omega=0$) can then be solved by using an iterative procedure (Rayleigh quotient) able to follow each identified branch to the desired value of angular speed.

![Figure 5. Absolute values $|\lambda|$ of forward natural frequencies as functions of $\dot{\omega}$ ( $\alpha=10, \sigma=2.933, \dot{N}=\dot{T}=0$). The dotted line indentify the asymptote $\lambda=2\omega$, the dashed curve identify the forward switch frequencies.](image)
5. Remarks on the switch frequencies

Recalling now the switch frequencies defined in Eq. (7), in the case of non-rotating unloaded Timoshenko beams they reduce to a unique value \( \lambda = \alpha^2 / \sqrt{\sigma} \). This critical value is sometimes referred to as cut-off frequency \([4, 5]\), while Eq. (7) generalizes its definition to the rotating and axially loaded case \([8]\).

If considering also a twisting moment, for realistic values of \( \hat{T} \) Eq. (7) can be considered a good approximation of the switch frequencies, and it still gives the exact frequency values in which no-total-deflection modal shapes occur \((\phi_0 = 0 \text{ and } \phi_\theta = \text{constant})\).

Clearly, the most influential parameter on the switch frequencies is the slenderness ratio \( \alpha \); however, \( \hat{\omega} \) can influence significantly \( \lambda_f \) and \( \lambda_b \) at high speed, while the effect of external loads in this case is of minor importance.

Figure 4 shows the two switch frequencies \((\lambda_0, \lambda_f)\) as functions of \( \hat{\omega} \) (grey dotted curves). From Eq. (7) it can be found that as \( \omega \to \infty \) then \( \lambda_0 \to 0 \) and \( \lambda_f \to +\infty \). The intersections among the curve defined by the two switch frequencies (as functions of \( \hat{\omega} \)), and the branches of the forward and backward eigenvalues (as functions of \( \hat{\omega} \) as well) can be easily found at least in the case \( T = 0 \) by introducing Eq. (7) into the characteristic equation for the exponents \( a_n \), yielding \( p_0 = 0 \) in Eq. (4) and consequently a quadratic (not a quartic) equation in \( \lambda \) (from here onwards the modal index is dropped).

Notice that in this case at the switch frequencies there is a unique non-zero value for \( |a| \). Equating the two roots with the switch frequencies, gives the two intersection values in implicit form:

\[
\hat{\omega}_\text{sw} = \pm \frac{\alpha}{2\sigma} \sqrt{\frac{|\sigma C - (1 + \sigma)\psi|^2}{|\sigma C - \psi|}}, \quad C = |a|^2 + \psi \tilde{N} \alpha^2
\]

where the \( \hat{\omega} \) forward intersection value is positive, with \( a = a(\lambda_f) \), and the backward one is negative, with \( a = a(\lambda_b) \). For large \( |a| \) the forward intersection value takes a simpler form:

\[
C \gg \psi \Rightarrow \hat{\omega}_\text{sw} = \frac{\alpha}{2\sqrt{\sigma}} \sqrt{C} \quad \omega \to \infty \Rightarrow \hat{\omega}_\text{sw} = k_\omega \lambda_f
\]

as expected (since the switch frequency curve is asymptotic to \( \lambda = k_\omega \hat{\omega} \), as shown in Figs. 4 and 5). All the branches of the first spectrum natural frequencies cross the the switch frequency curve twice (forward and backward), while those of the second spectrum always lay above it (the switch frequency curve is not a boundary between the two frequency spectra, it is a lower bound for the second frequency spectrum).

Therefore, at any given angular speed, all the forward eigenvalues smaller than the switch value (at that angular speed) belong to the first frequency spectrum. Above the switch value the frequencies of the two spectra overlap in some complicated fashion, however they can be identified in general by following the criterion given above.

As already noticed, at the switch frequencies the total deflection angles are zero, consequently the shear angles and the cross-section rotation angles are in counter-phase (equal and opposite if \( \psi = 1 \)), as it can be understood from the expression of the shear angle eigenfunctions:

\[
\phi_{xy} = \phi_s - \psi \phi_\theta = -\frac{\sigma \lambda^2}{2\alpha^2} \sum_{n=1}^{B_n} \frac{B_n}{a_n} e^{\lambda^2 t}
\]

On the other hand, as \( \omega \to \infty \), at the horizontal asymptotes the cross-section rotation angles become zero, hence the shear angles and the total deflection angles are in-phase (actually they coincide).

Consequently, increasing the angular speed and following a first spectrum forward branch which intersects the switch frequencies curve, change is observed in phase relations among cross-section rotation, shear and total slope.

As discussed in the literature, above the cut-off frequencies the results given by the Timoshenko beam theory become progressively less accurate. According to some authors, the whole second spectrum should be disregarded, and considered un-physical \([4, 5]\), in contrast with the results presented in \([6, 7]\). In any case, it should be noticed that the frequency range of validity of the model under analysis is reduced if considering small values of slenderness ratio \( \alpha \), and this is related to the fact that, beyond certain frequency limits, the assumption of planarity of cross-sections during deformation clearly becomes unrealistic.
6. Conclusions

New evidence of existence of a second frequency spectrum in the Timoshenko beam theory has been presented, together with a novel definition for its identification, possible if considering gyroscopic effects.

It has been found that a link between companion frequencies belonging to the first and second spectra is given by a peculiar asymptotic behaviour at high rotating speed. Each non-zero horizontal asymptote in Campbell diagrams represents a link between two pairs of eigenvalues, one pair at lower frequencies, one pair at higher frequencies. The latter, at zero angular speed and in particular cases of boundary conditions in which the characteristic equation factorizes (as in the case of simply supported ends), can be identified with the Timoshenko second spectrum. However, when considering a rotating shaft, as $\omega \to \infty$ the existence of non-zero horizontal asymptotes for any boundary conditions suggests a new way for defining and identifying the natural frequencies of the Timoshenko first and second spectra. All first spectrum backward eigenfrequencies tend to 0; all second spectrum forward eigenfrequencies tend to infinity, while the absolute value of each first spectrum forward eigenfrequency converges to the backward companion one belonging to the second spectrum. As a consequent result, it can also be stated that the whole second spectrum gives no contribution to the forward critical speeds. In parallel, the role of the so-called cut-off frequencies has been investigated, extending their definition to include the effects of gyroscopic moments and external loads.

REFERENCES