SELECTION OF NEAR FIELD ACOUSTIC HOLOGRAPHY REGULARIZATION BASED ON HELMHOLTZ LEAST SQUARE METHOD

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The selection of regularization methods is the key issue on Helmholtz equation-least squares (HELS)-based near-field acoustic holography (NAH). The characteristics of the truncated singular value decomposition (TSVD) method and standard Tikhonov regularization method were discussed respectively. The L-curve criterion and the generalized cross validation (GCV) method were both employed to attain the optimal regularization parameters for Tikhonov and TSVD regularization method. Furthermore, the maximum curvature of L-curve is used to determine the regularization parameter of the L-curve criterion. The numerical results that using the method of choosing the canonical parameter $\lambda$ to obtain the minimum value of the GCV function can effectively determine the optimal regularization parameter. In conformal planar near-field acoustic holography based on Helmholtz equation-least squares, reconstruction of single pulsatile ball near field radiation sound pressure value’s relative errors are less than 10% with the four combinations regularization method at 1000Hz frequency, the sound source location identification is exact. Finally, the reconstruction accuracy and robustness of the four combinatorial regularization methods are investigated.

Keywords: nah, regularization method, tikhonov method, generalized cross validation method, helmholtz equation-least squares

1. Introduction

Nearfield acoustic holography (NAH) is a very effective method for noise source identification and positioning. At present, the main reconstruction algorithms of NAH mainly include the Fourier transform and inverse transform [1], the boundary element method (BEM) [2] and the Helmholtz equation-least-square method squares (HELS) [3]. Each of these algorithms has its own advantages and disadvantages, and its essence is used to construct the vibrational acoustic transmission relationship between the sound pressure and the sound source, and then the vibrational acoustic characteristics of the sound source are inverted from the measured holographic surface acoustic pressure. Due to the error in the measurement of the holographic surface is inevitable, the solution obtained by the direct inversion is often meaningless due to the perturbation of the measurement error. Therefore, it is usually necessary to regularize the effect of measurement error on the reconstruction solution in the process of inverse solving.

Regularization methods are divided into direct regularization method and iterative regularization method. The direct regularization methods commonly is used in NAH including truncated singular value decomposition (TSVD) [4-5] and Tikhonov regularization [6-7]. In addition, iterative regularization methods are also used in NAH, such as the Landweber iterative regularization method [7] and the conjugate gradients (CG) method. So far, there is no optimal regularization method to adapt to any problem, so it is very meaningful to compare different regularization methods.

After determining the regularization method, the selection of regularization parameters will become crucial, which is the most sensitive factor affecting the reconstruction solution. At present, the
regularization parameter selection method that depends on the measurement error information has a dispersion principle method [8]. The parameter selection method that does not use the error information mainly includes the L curve criterion [9], the generalized cross validation (GCV) [10].

Since NAH based on the least square method of Helmholtz equation is a relatively hot method of sound source localization at present, the near-field acoustic holographic mathematical model based on the least square method of Helmholtz equation is established in this paper. Based on this, the methods of truncated singular value method and Tikhonov regularization are introduced, and the basic principle and characteristics of L-curve criterion and generalized cross-test method are described. By comparing the numerical simulation results of four different combinations of these four regularization methods and two regularization parameter selection methods in conformal planar near-field acoustic holography. The reconstruction accuracy and adaptability of different combination regularization methods are studied in order to provide a basis for practical application.

2. Spherical wave expansions based NAH method

The general solution of the Helmholtz equation in the spherical coordinate system is

\[ p(\gamma, \theta, \phi; w) = \sum_{n=0}^{\infty} \sum_{l=-n}^{n} \{ C_n h_n^{(1)}(kr) + d_n h_n^{(2)}(kr) \} Y_l^m(\theta, \phi) \]  

(1)

Where \( h_n^{(1)}(kr) \) and \( h_n^{(2)}(kr) \) are spherical Hankel functions of the first and second kind, respectively, \( Y_l^m(\theta, \phi) \) represents the spherical harmonic function, \( k \) is the acoustic wavenumber, \( C_n \) and \( d_n \) are the coefficients which need to be determined. In the free field, the acoustic field can be expressed using spherical Hankel functions of the first kind only as

\[ p(r, \theta, \phi; w) = \sum_{n=0}^{\infty} \sum_{l=-n}^{n} C_n h_n^{(1)}(kr) Y_l^m(\theta, \phi) \]  

(2)

Base on this formulation, Wu et al [15] proposed a method to approximate the solution of an acoustic field by using a finite expansion of the spherical wave functions, where the origin of the expansions are set at the geometric center of the source. In this method, the reconstruction formulation can be expressed in the matrix form as

\[ P(x) = G_p(x|x_m)P(x_m) \]  

(3)

Where \( P(x) \) is a column vector of unknown acoustic pressures on the reconstruction surface, \( P(x_m) \) is a column vector of measured pressures on the hologram surface, and \( G_p(x|x_m) \) is the transfer matrix relating the measured acoustic pressures to the reconstructed acoustic pressures at any location in the field or on the source surface, which is given by

\[ G_p(x|x_m) = \Psi(x)\Psi(x_m)^+ \]  

(4)

Where \( \Psi(x_m)^+ \) is a pseudo-inverse matrix of \( \Psi(x_m) \):

\[ \Psi(x_m)^+ = [\Psi(x_m)^H\Psi(x_m)]^{-1}\Psi(x_m)^H \]  

(5)

Where the superscript \( H \) implies a conjugate transpose of the matrix. Note that the terms making up \( \Psi \) are the particular solutions to the Helmholtz equation, which can be written as

\[ \psi_j(r, \theta, \phi; w) \equiv \psi_m(r, \theta, \phi; w) = h_n^{(1)}(kr)Y_l^m(\theta, \phi) \]  

(6)

Where \( j = n^2 + n + 1 + 1 \).

This method can be used to reconstruct the acoustic field between the source and the hologram surface, including the source surface, as well as to predict the radiation field of the source. However, if there exist sources on both sides of the hologram surface, this method isn’t applicable to reconstruct the acoustic field of the source, because the acoustic field is no longer a free field.
3. Regularization method

In practical problems, \( \Psi(x_m) \) are often ill-posed matrices, which is ill-conditioned matrices. Regularization is a common measure to solve the problem of ill-posedness, the direct regularization methods commonly is used in NAH include truncated singular value decom-position (TSVD) and Tikhonov regularization. In addition, iterative regularization methods are also used in NAH, such as the Landweber iterative regularization method and the conjugate gradients (CG) method. In this article, truncated singular value method and Tikhonov regularization method are only introduced.

3.1 truncated singular value decom-position (TSVD)

Regularization solution can be expressed as

\[
(x)_{reg} = \sum_{i=1}^{N} f_i \frac{u_i^T p}{\sigma_i} v_i
\]

(7)

The basic principle of the TSVD method is to set a threshold to directly take the value of the \( N \) singular values which are smaller than the threshold to zero, it can remove the influence of the small singular values on the reconstructed solution. The threshold is usually achieved by a scaling factor that sets it to zero if \( \sigma_i < \hat{\sigma}_i \). The corresponding filter coefficient expression:

\[
f_i = \begin{cases} 1, & \sigma_i \geq \hat{\sigma}_1 \\ 0, & \sigma_i < \hat{\sigma}_1 \end{cases}
\]

(8)

Where \( \hat{\sigma} \) represents the truncation ratio coefficient.

At this time, regularized reconstruction solution can be expressed as

\[
(x)_{reg} = \sum_{i=1}^{k} \frac{u_i^T p}{\sigma_i} v_i
\]

(9)

Where \( k \) represents the singular value of the cut-off point.

How to choose the cut-off point \( k \) is the key to the effective implementation of this method. The value of \( k \) is too large to be chosen, it will cause that a large amount of useful information in the sound field will be lost, while the choice is too small to interfere with the measurement error. To this end, attempts will be made to solve the optimal cut-off point \( k_{opt} \) by using the following L curve criterion and GCV method.

3.2 Tikhonov regularization (TR)

Tikhonov regularization method is one of the most popular methods. It is based on the recognition that there is always a large number of discrete values in the vector space under morbid conditions. Therefore, based on the least squares solution, additional constraints are imposed to minimize the solution vector norm. Generalized Tikhonov Regularization Solution is expressed as

\[
(x)_{reg} = \min \{ \|Ax - b\|^2 + \lambda^2 \|L(x - \hat{x})\|^2 \}
\]

(10)

Where \( \lambda \) is the regularization parameter; \( L \) is the regularization matrix; \( \hat{x} \) is the initial estimate of the solution vector.

When \( L = I_n \), \( \hat{x} = 0 \). formula (10) is called the standard Tikhonov regularization method. Then the standard Tikhonov regularization solution is

\[
(x)_{reg} = (A^T A + \lambda^2 I_n)^{-1} A^T b = \sum_{i=1}^{N} \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \frac{u_i^T p}{\sigma_i} v_i
\]

(11)

According to Eq. (11), the filter coefficient of the standard Tikhonov regularization method is

\[
f_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \approx \begin{cases} 1, & \sigma_i \gg \lambda \\ \sigma_i^2 / \lambda^2, & \sigma_i \ll \lambda \end{cases}
\]

(12)
Comparing Eqs. (7) and (12), we find that the TSVD method and the standard Tikhonov method are approximate in nature, but only to a certain degree of difference in reducing the influence of small singular value terms. The TSVD method completely removes the influence of small singular value terms on the solution, while the standard Tikhonov method only weakens this effect and has not been completely removed. Therefore, when the small singular value item contributes a small amount to the solution, the direct removal will help to obtain a satisfactory solution. Theoretically, the solution obtained by TSVD in this case is superior to the standard Tikhonov regularization method. However, when the small singular value item contributes a lot to the solution, TSVD will lose some useful information, resulting in excessive regularization error.

4. Selection methods for regularization parameter

The reconstruction solution based on the Helmholtz method of near-field acoustic holography is very sensitive to the selection of regularization parameters. The optimal regularization parameter must be able to strike a good balance between regularization error and measurement error. At present, the regularization parameter selection methods commonly used are the L curve criterion and the generalized cross-test method.

4.1 L curve method

The L curve criterion is to obtain the L-shaped relation curve between the residual norm $\|Ax_{\text{reg}} - b\|_2$ and the regularization norm $\|x_{\text{reg}}\|_2$ according to the reconstructed solution under different regularization parameters through a two-dimensional chart. According to the requirement of the balance error of the optimal regularization parameter, the corner of the L curve is the optimal value of the regularization parameter. For the continuous regularization parameter $\lambda$, the corner point is determined by solving $\left(\log \|Ax_{\text{reg}} - b\|_2, \log \|x_{\text{reg}}\|_2\right)$.

defined as

$$
\eta = \|x_{\text{reg}}\|_2^2 \\
\rho = \|Ax_{\text{reg}} - b\|_2^2
$$

make

$$
\begin{align*}
\hat{\eta} & = \log \eta \\
\hat{\rho} & = \log \rho
\end{align*}
$$

(13)

Then the curvature of the L curve [12] is

$$
\kappa = 2 \frac{\hat{\rho}'\hat{\eta}'' - \hat{\rho}''\hat{\eta}'}{((\hat{\rho}')^2 + (\hat{\eta}')^2)^{3/2}}
$$

(14)

where $\hat{\rho}'$, $\hat{\rho}''$, $\hat{\eta}'$ and $\hat{\eta}''$ represent the first and second order differential of $\rho$ and $\eta$, respectively. due to

$$
\begin{align*}
\hat{\eta}' & = \frac{\eta'}{\eta} \\
\hat{\rho} & = \frac{\rho'}{\rho} \\
\rho' & = -\lambda^2 \eta'
\end{align*}
$$

(15)

Substituting (14), we get

$$
\kappa = 2 \frac{\eta \rho \lambda^2 \eta' \rho + 2\lambda \eta \rho + \lambda^4 \eta'}{(\lambda^2 \eta^2 + \rho^2)^{3/2}}
$$

(16)

where $\eta' = -\lambda^2 \sum_{i=1}^{N} (1 - f_i) \eta_i^2 \frac{u_i \bar{b}}{\sigma_i}$. 


The optimal regularization parameter $\lambda_{opt}$ corresponding to the maximum curvature can be obtained from (16). For the discrete regularization parameter $k$, by substituting $\lambda = \gamma_k$ into (16), the optimal truncation point $k_{opt}$ corresponding to the maximum curvature can be obtained.

### 4.2 Generalized cross-test method

The basic idea of near-field acoustic holographic generalized cross-test is to assume that any one holographic measurement point does not participate in the reconstruction calculation, then the reconstructed solution is obtained from the remaining measurement points should be able to predict the sound pressure at the measurement point accurately. The generalized cross-test method is equivalent to solving the minimum GCV function [14].

$$G(\lambda) = \frac{\|Ax_{reg} - b\|_2^2}{\text{trace}(I - AA^T)}$$

where $A^T$ is the pseudo-inverse matrix after the transfer matrix $A$ is regularized, through $A^T$ can calculate the regularization solution, $x_{reg} = A^Tb$, $\text{trace}(\cdot)$ represents the trace of the matrix, that is, the sum of the elements on the diagonal of the matrix. The numerator of $G(\lambda)$ is the regularized residual. The $\lambda$ that minimizes $G(\lambda)$ is the regularization parameter of GCV.

### 5. Numerical simulation

The above regularization methods and regularization parameter selection methods were divided into four groups to examine: (I) standard Tikhonov regularization method + L-curve criterion, (II) standard Tikhonov regularization method + GCV method, (III) TSVD method + GCV method. (IV) TSVD method + L curve criterion.

![Figure 2: Schematic of the acoustic imaging of a pulsating ball](image)

In Figure 2, the object of study is a single pulsating sphere. The radius of the pulsation sphere is $r_0 = 0.1m$, and the center is located at the origin of the coordinates. The vibration frequency is $f = 1000Hz$, and the amplitude of surface normal vibration velocity is $v_0 = 1 m/s$, air density $\rho = 1.29kg/m^3$, the speed of propagation of sound waves in the air $c = 340m/s$. According to the theoretical formula, the radiated sound pressure of the pulsating sphere at any point $x$ is:

$$p(x) = \frac{i\rho c h r_0^2 k}{r(1 + ikr_0)} e^{i(wt - kr)}$$

Where $r = |x|$ is the distance from the field point to the origin of the coordinate system, $k = w / c$ is the wave number. A more common rectangular grid array is used as the holographic measurement surface $H$, uniform distribution above $6 \times 6$ measuring points, the network in the $x$ direction and $y$ direction spacing are 0.05m. The reconstructed surface $R$ is parallel to the holographic surface $H$, which has a size of $1.5m \times 1.5m$ and a reconstruction interval of 0.05 m. The centers of the holographic plane $H$ and the reconstructed plane $R$ are both located on the $z$-axis. The measured distance $z_H = 0.18$ and the reconstruction distance $z_R = 0.15 m$. Considering that measurement error will inevitably occur in the actual measurement, Gaussian white noise is added to the theoretical sound pressure measured by the holographic plane through the matlab software so that the signal-to-noise ratio (SNR) of the final measured sound pressure is 30 dB.

The accuracy of the sound field reconstruction directly affects the correctness of the acoustic imaging results. The evaluation indexes are mainly the relative errors $e\,$, and the root mean square...
error $e_{\text{rms}}$. Among them, the reconstructed surface reconstruction point $x$ sound field reconstruction relative error $e(x)$ is

$$e(x) = \frac{|p(x) - \bar{p}(x)|}{|\bar{p}(x)|} \times 100\% \quad (19)$$

In the formula, $p(x)$ and $\bar{p}(x)$ represent the reconstruction sound pressure at the reconstruction point and the theoretical sound pressure are calculated by the above formula respectively. In order to quantitatively describe the error level of imaging from an overall perspective, so as to compare the reconstruction effects which are obtained under different regularization combinations, the root mean square error $e_{\text{rms}}$ of the sound field reconstruction is defined as

$$e_{\text{rms}} = \frac{\|p_{R} - \bar{p}_{R}\|_2}{\|\bar{p}_{R}\|} \times 100\% \quad (20)$$

In the formula, $p_{R}$ and $\bar{p}_{R}$ are reconstructed sound pressure vectors and theoretical vectors.

In order to compare the effects of regularization, the reconstructed solution to the problem is obtained without any regularization measures, the comparison with the exact solution is shown in Figure 3. At this time, due to the addition of Gaussian white noise measurement error (signal-to-noise ratio 30 dB), a large reconstruction solution relative error (37,882,752%) is caused, which makes the reconstructed solution completely free of distribution trends and loses the reconstruction value.

Using a combination of regularization methods (I), (II), (III) and (IV) respectively, the above problems are regularized, and the optimal regularization parameters are obtained as shown in Fig.4 ~ Fig.7. The reconstructed surface acoustic pressure distribution is shown in Figure 8. Table 1 shows the relative error and root-mean-square error of the reconstructed solutions which are generated by these four combinations of regularization methods.
Table 1: Relative reconstruction error of four regularization methods

<table>
<thead>
<tr>
<th>Group Number</th>
<th>Tikhonov L-curve</th>
<th>Tikhonov GCV</th>
<th>TSVD GCV</th>
<th>TSVD L-curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative error 𝑒(𝑥) / %</td>
<td>6.69</td>
<td>5.90</td>
<td>7.89</td>
<td>8.41</td>
</tr>
<tr>
<td>Root mean square error 𝑒_{rms}</td>
<td>7.92</td>
<td>6.93</td>
<td>9.12</td>
<td>9.54</td>
</tr>
</tbody>
</table>

From the comparison of the sound pressure distribution and the exact solution of the reconstructed surface are shown in Fig. 8, it can be known that the four regularization methods can effectively identify the acoustic characteristics of the sound source surface. It can be seen that the resolution of the conformal plane acoustic holography at the frequency of 1000 Hz is met by the combination of the NAH based on the Helmholtz least square method and the above four regularization methods.

The difference between the reconstructed solution cloud maps which are obtained by the four regularization methods are shown in Fig. 8 is not very clear. However, by comparing the relative reconstruction errors in Table 1, it can be obtained that the reconstruction accuracy for the above NAH problem is ranked from high to low: combination (II)>combination (I)>combination (IV)>combination (III). At the same time, it can also be seen that in the numerical example, for the same regularization method (Tikhonov method or TSVD method), the GCV method has a higher accuracy of the reconstructed solution than the L-curve criterion. When using the same regularization parameter selection method, the Tikhonov method has a higher accuracy than the standard TSVD regularization method.

For the standard Tikhonov regularization method, comparing Fig. 4 and Fig. 5, it can be seen that a relatively weaker regularization effect is produced by L-curve than the GCV method because 𝜆_{L-curve}=3.6e-8 is smaller than 𝜆_{GCV} = 4.6e-4, which is fewer small singular values are filtered. In the TSVD regularization method, comparing Figure 6 and Figure 7, we can see that because 𝜅_{L-curve} = 6
is smaller than $k_{\text{GCV}} = 16$, the L-curve has a relatively stronger regularization effect on the reconstructed solution than the GCV method, which truncates more small singular values. Therefore, the two regularization parameter selection methods have different regularization trends in different regularization methods.

6. Conclusions

(1) In the NAH based on the Helmholtz least square method, the reconstruction solution is very sensitive to sound pressure measurement errors. In the actual sound field, there must be background noise and interference sources, regular filtering must be performed in order to obtain a reasonable reconstruction solution.

(2) The numerical results show that when the signal-to-noise ratio is 30dB, the four normalization methods of NAH combined with the error at 1000 Hz are all less than 10%, and the resolution is satisfactory based on the Helmholtz least squares method.

(3) For the same regularization method, the generalized cross-validation (GCV) method is more accurate than the L-curve method in obtaining near-field acoustic holographic reconstruction solutions. The regularization trends are obtained by the two regularization parameter selection methods in different regularization methods are not the same.

(4) In the above near field acoustic holography problem, when the Tikhonov regularization method uses the L-curve method and the GCV function method to solve the optimal regularization parameters, the Tikhonov method can obtain a more satisfactory reconstruction result than the TSVD regularization method.

(5) In the plane-conformal near-field acoustic holography, considering the reconstruction accuracy and robustness, the combination regularization method of Tikhonov method and GCV parameters is the best choice.

REFERENCES


