Studies on elastic waves interacting with imperfect interfaces are relevant in many practical applications, such as the acoustic testing for detection of damages and cracks in solids. The present analysis considers a contact problem between an elastic half-space and a restrained rigid surface. A simplified mathematical model of the imperfect interface is established in the presence of bulk wave excitation. We start with an initial assumption that the half-space is perfectly bonded to the rigid surface. The different wave fields at the interface are then corrected by a linear superposition of secondary solutions to satisfy the conditions of interfacial imperfection. Based on a direct linear relation between the interfacial slip and traction, a simple linear spring model is used to model the imperfection. Also, at such an interface, the available traction fields may cause formation of gap in the form of a globule. Interestingly, due to the gap and imperfection at the interface, the dynamics of the elastic medium involves both inequality and equality constraints. The various interfacial fields are thus interrupted at either or both the ends of a gap. The effects of degree of imperfection, excitation parameters and material properties on the interface dynamics have been studied and discussed.

Keywords: bulk wave, interface, imperfection, gap

1. Introduction

Propagation of waves in continuous media and its energetics are often used in many engineering applications. For example, in acoustic measurement of defects in structures, in seismology etc. There have been various studies on elastic waves in the context of planar boundaries [1]. However, a perfectly bonded interface has considered and analyzed in most of these studies [2-4]. In this analysis, the interface is assumed to be bonded imperfectly with a restrained rigid surface.

Fundamental problems of imperfect interface have been considered in various structural investigation for different purposes. Murty [5] has investigated such interfaces in the presence of elastic wave fields. In the absence of any interfacial gap, an analysis on wave energetics is presented in his study by considering the possible boundary conditions at the interface. An analytical approach for the nonuniform and homogeneous boundaries has been proposed in Ref. [7], where the such boundaries are considered as linear interfacial slip.

This work deals with the analysis of interaction of wave with a imperfect interface. The method of wave field correction is used to formulate the problem as in Refs. [8] and [9]. At the interface, the linear relations between the slips and the stresses are considered. With the available stress fields and imperfect bonding, a close gap is form at the interface in the form of a globule. Such complex interface leads to comparatively complicated analysis in the sense that the boundary involves both inequality and equality constrains. The effect of external loads and degree of imperfection on the state of stress and slip at the interface is studied and discussed.
2. **Formulation**

We consider an elastic half-space (homogeneous and isotropic) of density $\rho$ in contact with a restrained rigid surface. An incident P-wave of frequency $\omega$ strikes the imperfect interface, as shown in Fig. 1. The incident wave is partially reflected as P- and SV-waves in the half-space. Hence, the total displacement in terms of potentials is given as

$$\phi = \Phi_P \exp [i (K_P \cdot e - \omega t)] + \Phi_2 \exp [i (K_P \cdot e - \omega t)],$$

$$\psi = \Psi_S \exp [i (K_S \cdot e - \omega t)],$$

where $\Phi_P$ is the amplitude of incident P-wave potential, $\Phi_2$ and $\Psi_S$ are, respectively, the amplitudes of reflected P- and SV-waves potentials, and $K_P = k_P \sin \theta_P e_1 - 2j(k_P \cos \theta_P e_2)$, $j = 1, 2$ and $K_S = k_S (\sin \theta_S e_1 - \cos \theta_S e_2)$, are, respectively, the propagation vectors of P- and SV-waves.

The displacement and stress fields corresponding to potentials given by Eq. (1) and Eq. (2) are,

$$u_1 = \partial \phi/\partial x_1 + \partial \psi_3/\partial x_2,$$

$$u_2 = \partial \phi/\partial x_2 - \partial \psi_3/\partial x_1,$$

where $\lambda$ and $\mu$ are the Lamé parameters. From the requirement of vanishing displacement boundary at $x_2 = 0$ for the initially assumed perfectly bonded interface, we have

$$u_1|_{x_2=0} = 0, \quad \Rightarrow \quad [\partial \phi/\partial x_1 + \partial \psi_3/\partial x_2]|_{x_2=0} = 0,$$

$$u_2|_{x_2=0} = 0, \quad \Rightarrow \quad [\partial \phi/\partial x_2 - \partial \psi_3/\partial x_1]|_{x_2=0} = 0.$$

The above conditions are identically valid over all $x_1$ and $t$ values if

$$k_P \sin \theta_P = k_P \sin \theta_P = k_S \sin \theta_S = k, \quad \text{and} \quad c_P k_P = c_P k_P = c_S k_S = \omega.$$

Here, $c_P$ and $c_S$ are the P- and SV-wave velocities, respectively. Thus, for P-wave, we have $k_P = k_P = k_P$ and $\theta_P = \theta_P = \theta_P$. Now, solving Eqs. (3) and (4) for $\Phi_2$ and $\Psi_S$ using the boundary conditions (7) and (8), and substituting in Eqs. (5) and (6), we get the stresses at the boundary as

$$\sigma_{22} = n \cos(kx_1 - \omega t), \quad \sigma_{12} = s \cos(kx_1 - \omega t),$$

where $n$ and $s$ are constants.
where
\[ n = -k_p^2 (\lambda + \mu) (\Phi_{P1} + \Phi_{P2}) - k_p^2 \mu (\Phi_{P1} + \Phi_{P2}) \cos 2\theta - k_S^2 \mu \Psi_S \sin 2\theta, \]
\[ s = -k_S^2 \mu \Psi_S \cos 2\theta - k_p^2 \mu (\Phi_{P1} - \Phi_{P2}) \sin 2\theta, \]
which depend on the medium and wave parameters.

In order to have imperfect interface conditions, we introduce secondary wave fields in the above solutions. The wave potentials for the secondary fields are written as
\[
\vec{\phi} = \vec{\Phi}_{P2} \exp[i (K_{P2} \cdot \mathbf{e} - \omega t)], \quad \text{and} \quad \vec{\Psi} = \vec{\Psi}_S \exp[i (K_S \cdot \mathbf{e} - \omega t)],
\]
where \( \vec{\Phi}_{P2} = (A_1 + iB_1) \) and \( \vec{\Psi}_S (\vec{\Psi}_S = \vec{\Psi}_{SxS}) = (A_2 + iB_2) \) are the complex amplitudes. Here, we assume that the Snell’s law still hold true without any harmonics generation.

Now, the stresses are calculated corresponding to potentials (11) and (12). Hence, using Eqs. (9) and (10) the total interfacial stress field is written as
\[
N(r) = -n_0 + n \cos r + (-\lambda k_p^2 - 2\mu k_p^2 + 2\mu k^2) \left( A_1 \cos r - B_1 \sin r \right) \]
\[
+ \left[ -2\mu k \left( k_S^2 - k^2 \right)^{1/2} \right] \left( A_2 \cos r - B_2 \sin r \right),
\]
\[
S(r) = s_0 + s \cos r + \left[ 2\mu k \left( k_p^2 - k^2 \right)^{1/2} \right] \left( A_1 \cos r - B_1 \sin r \right) \]
\[
+ \left( -\mu k_S^2 + 2\mu k^2 \right) \left( A_2 \cos r - B_2 \sin r \right),
\]
where \( r = (krx - \omega t) \), and \( n_0 \) and \( s_0 \) are the external loads. In a similar way, the slip at the interface with respect to rigid surface is calculated as
\[
v_1(r) = -\hat{u}_1 = -k^2 c \left( A_1 \cos r - B_1 \sin r \right) + \left[ k c \left( k_S^2 - k^2 \right)^{1/2} \right] \left( A_2 \cos r - B_2 \sin r \right), \]
\[
v_2(r) = -\hat{u}_2 = \left[ k c \left( k_p^2 - k^2 \right)^{1/2} \right] \left( A_1 \cos r - B_1 \sin r \right) + k^2 c \left( A_2 \cos r - B_2 \sin r \right),
\]
where \( c = \omega / k \) is the phase velocity in horizontal direction.

### 2.1 Interface boundary conditions

In the gap region, there is zero normal stress and zero shear stress \( (N(r), S(r) = 0) \), along with the inequality condition \( G(r) \geq 0 \). Thus, using Eqs. (13)-(16), we obtain the expression of gap by integrating normal slip \( v_2(r) \) with respect to \( r \) as
\[
G(r) = -\left[ k c \left( M_{12}M_{21} - M_{11}M_{22} \right) \right]^{-1} \left[ s_0 M_{11}r + n_0 M_{21}r + (s M_{11} - n M_{21}) \sin r + L \right],
\]
where the matrix \( \mathbf{M} \) is given by
\[
\mathbf{M} = \begin{bmatrix}
-\lambda k_p^2 - 2\mu k_p^2 + 2\mu k^2 & -2\mu k \left( k_S^2 - k^2 \right)^{1/2} \\
2\mu k \left( k_p^2 - k^2 \right)^{1/2} & -\mu k_S^2 + 2\mu k^2
\end{bmatrix} \begin{bmatrix}
-k^2 c & k c \left( k_S^2 - k^2 \right)^{1/2} \\
k c \left( k_p^2 - k^2 \right)^{1/2} & k^2 c
\end{bmatrix}^{-1}.
\]
The coordinates of gap \( (a, b) \) can be found from zero normal slip condition at the boundaries, and by using constant of integration \( L \) in Eq. (17).

In the slip region \( (N(r) \leq 0) \), an imperfect interface boundary can be characterized by a simple linear spring model. Hence, the corresponding mathematical conditions for the slip region are written as
\[
\phi_N N(r) = (1 - \phi_N) Z v_2(r), \]
\[
\phi_S S(r) = (1 - \phi_S) Z v_1(r),
\]
where $\phi_N$ and $\phi_S$ are the appropriate parameters representing the degree of imperfection in normal and transverse direction, respectively. The parameters vary in the range (0, 1), and the limiting values 0 and 1 describe the bounded and the unbounded interface cases, respectively. $Z$ is a finite constant having the dimension of specific acoustic impedance.

Eliminating complex amplitudes for secondary fields between Eqs. (13–16), and subsequently using boundary conditions, we can write expressions for interfacial slip as

$$ v_1 (r) = \begin{cases} - \frac{M_{22} (n_0 - n \cos r) - M_{12} (-s_0 - s \cos r)} {M_{21} M_{12} - M_{11} M_{22}}, & \text{if gap} \\ - \frac{M_{22} (n_0 - n \cos r) - (M_{12} - \frac{1-\phi_N}{\phi_N} Z)(-s_0 - s \cos r)} {M_{21} - \frac{1-\phi_S}{\phi_S} Z} (M_{12} - \frac{1-\phi_N}{\phi_N} Z) - M_{11} M_{22}, & \text{otherwise} \end{cases} \quad (20) $$

$$ v_2 (r) = \begin{cases} - \frac{M_{21} (n_0 - n \cos r) + M_{11} (-s_0 - s \cos r)} {M_{21} M_{12} - M_{11} M_{22}}, & \text{if gap} \\ - \frac{M_{21} (n_0 - n \cos r) + (M_{12} - \frac{1-\phi_N}{\phi_N} Z)(n_0 - n \cos r) + M_{11} (-s_0 - s \cos r)} {M_{21} - \frac{1-\phi_S}{\phi_S} Z} (M_{12} - \frac{1-\phi_N}{\phi_N} Z) - M_{11} M_{22}, & \text{otherwise} \end{cases} \quad (21) $$

Similarly, from the above expressions, one can obtain the state of stress at the interface using Eqs. (13) and (14).

### 3. Results and discussion

In order to understand the effect of imperfection on the wave dynamics, the numerical results of the problem considered are presented and discussed in this section. We consider the half-space is made of aluminum with density $\rho = 2700 \text{ kg m}^{-3}$, and wave speeds as $c_P = 6300 \text{ m s}^{-1}$ and $c_S = 3100 \text{ m s}^{-1}$ [2]. In the half-space, an elastic P-wave is incoming towards the interface and partially reflected as P- and SV-wave. We begin by considering the gap formation at the interface. Subsequently, the imperfect boundary characterized by a linear spring model is analyzed at the slip region of the interface. We study the effect of external loads and imperfection parameters on the state of stress and slip at the interface. Here, the solution is having $2\pi$ periodicity in $r$.

(a) $n_0/n = 0.9, s_0/s = 0.1, \phi_N = 0.3, \phi_S = 0.3$

(b) $n_0/n = 0.8, s_0/s = 0.1, \phi_N = 0.6, \phi_S = 0.6$

![Figure 2: Non-dimensional tangential slip distribution.](image)

The interfacial slip distributions along the translating coordinate $r$ are shown in Figs. 2 and 3 for a fixed incident angle $\theta_{P1} = 30^\circ$ and parameter $Z = 1 \text{ MPa s m}^{-1}$. The distributions of non-dimensional quantities $v_1/\Phi_{P1} k^2 c$ and $v_2/\Phi_{P1} k^2 c$ are plotted for certain values of external loads and imperfection parameters, which are indicated in the figures. It is noticed that, due to the effects of
Figure 3: Non-dimensional normal slip distribution.

external loads and the imperfection parameters, the tangential slips have jump at the boundaries of gap $r = a$ and $r = b$. However, the effects are quite small for the normal slip, and hence the distributions are relatively continuous. In various plots, these jumps are marked by the dotted lines.

Figure 4: Non-dimensional normal stress distribution.

The state of stress for certain values of external loads and imperfection parameters are presented in Figs. 4 and 5. Changing the parameters $\Phi_N$ and $\Phi_S$ affects the stress variation significantly. For
a smooth or a frictional boundary, a partially similar behavior has also been noticed and discussed previously by Comninou and Dundurs [8, 9]. In all the cases, a clear jump discontinuity is observed at the gap boundary \( r = (a + 2m\pi) \), where \( m = 0, 1, 2, \ldots, \infty \). However, for the case of normal stress, the discontinuity is not observed at the second boundary. In particular, the condition \( N(r) \leq 0 \) also satisfy in the slip region. Further, the fields in the gap are independent of the parameters \( \Phi_N \) and \( \Phi_S \).

4. Conclusions

In this paper, an imperfect boundary between an elastic half-space and a rigid surface with an incident P-wave in the half-space has been considered. In the formulation, the wave field correction is introduced to take care of the imperfect interface. The linear spring model has been used in the analysis. Also, the interface is considered to be separated locally in the form of a globule. The effects of the external loads and imperfection on the interface behavior has been studied through a numerical example. It is observed that, the various fields at the interface are interrupted at either or both the ends of a gap, which are mainly depend on the degree of imperfection. The formulation can be extended further to analyze the wave energetics with existence of such interfaces.

REFERENCES