OPERATIONAL MODAL ANALYSIS USING THE FREQUENCY-SCALE DOMAIN DECOMPOSITION TECHNIQUE

Thien-Phu Le
LMEE, Université d’Evry Val d’Essonne. 40 rue du Pelvoux, 91020 Evry cedex, France
e-mail: thienphu.le@univ-evry.fr

Ambient vibration testing is the most convenient technique of dynamic testing for real structures because of several advantages: real boundary conditions, ambient excitation, and continuous use of the structures. From ambient vibration responses, modal parameters involving natural frequencies, damping ratios and mode shapes are identified by operation modal analysis that can be classified by signal processing domain (e.g., time, frequency and time-frequency).

In this paper, a new operational modal analysis technique based on the frequency-scale domain decomposition is proposed. A direct link between modal parameters and coordinates of local maxima in the frequency-scale plane of wavelet transform of the power spectral density of responses, is first established. Based on the link, modal parameters are then identified and the identification process is facilitated using the singular value decomposition algorithm. The Cauchy and Morlet mother wavelets, popular in wavelet transform processing are investigated. Analytical results are presented and a modal identification practical procedure from ambient vibration responses is finally proposed. The validity of the proposition is verified with a numerical example and an experimental test. In comparison to other wavelet-based methods, the proposed technique allows to extract modal parameters without the ridges extraction step of wavelet transform.

Keywords: Modal identification, Ambient Vibration Responses, Cauchy Wavelet, Morlet Wavelet, Frequency-Scale Domain Decomposition.

1. Introduction

A modal model involving natural frequencies, damping ratios and mode shapes called modal parameters, are always appreciated to represent a mechanical system. The modal parameters can be identified from measured responses collected in dynamic tests. The most convenient technique for real structures, is ambient vibration testing due to its advantages of low cost, real boundary conditions and continuous service [1].

Excitation from operational condition not measured, is always assumed to be a Gaussian white noise process, and modal identification from responses only is named Operational Modal Analysis (OMA). Several methods are proposed for OMA. They can be in time domain [2, 3], frequency domain [4, 5] or time-frequency domain [6, 7, 8, 9].

Using wavelet based methods in time-frequency domain, modal parameters are identified after an extraction step of ridges/skeleton of the wavelet transform [10, 11, 12]. Recently, Le and Argoul [13] and Le [14] propose the Frequency-Scale Domain Decomposition (FSDD) method for modal identification. The authors showed that modal parameters can be directly obtained from coordinates of stationary points in the frequency-scale plane.
The objective of this paper is to investigate the FSDD method with both mother wavelets of Morlet and Cauchy. After a brief background of ambient responses and wavelet transform in Section 2, the development of the FSDD method is presented in Section 3. Based on analytical formulation, a step-by-step practical procedure of the FSDD method is proposed. Section 4 is devoted to validity tests consisting of a numerical example and an experimental test. Finally some conclusions are given in Section 5.

2. Ambient vibration responses and continuous wavelet transform

2.1 Ambient vibration responses

Ambient excitation is always assumed to be a zero mean Gaussian white noise process. Under the excitation, responses \( x(t) \) of a \( N \) degrees-of-freedom linear mechanical system, is a zero mean Gaussian noise process [15, 16, 17].

The correlation function matrix of responses \( R_{xx}(\tau) \) and its Fourier transform matrix, called the power spectral density matrix \( \hat{R}_{xx}(\omega) \), are expressed by [9]:

\[
\begin{align*}
R_{xx}^+(\tau) &= \sum_{k=1}^{N} \phi_k e^{\lambda_k \tau} q_k^T + \bar{\phi}_k e^{\bar{\lambda}_k \tau} \bar{q}_k^T = \sum_{k=1}^{N} \left[ A_k e^{\lambda_k \tau} + \bar{A}_k e^{\bar{\lambda}_k \tau} \right] \quad \text{for } \tau \geq 0 \\
\hat{R}_{xx}^+(\omega) &= \sum_{k=1}^{N} \frac{\phi_k}{i\omega - \lambda_k} q_k^T + \frac{\bar{\phi}_k}{i\omega - \bar{\lambda}_k} \bar{q}_k^T = \sum_{k=1}^{N} \frac{A_k}{i\omega - \lambda_k} + \frac{\bar{A}_k}{i\omega - \bar{\lambda}_k} \\
R_{xx}^-(\tau) &= \sum_{k=1}^{N} q_k e^{-\lambda_k \tau} \phi_k^T + \bar{q}_k e^{-\bar{\lambda}_k \tau} \bar{\phi}_k^T = \sum_{k=1}^{N} \left[ A_k^T e^{-\lambda_k \tau} + \bar{A}_k^T e^{-\bar{\lambda}_k \tau} \right] \quad \text{for } \tau < 0 \\
\hat{R}_{xx}^-(\omega) &= \sum_{k=1}^{N} \frac{q_k}{-i\omega - \lambda_k} \phi_k^T + \frac{\bar{q}_k}{-i\omega - \bar{\lambda}_k} \bar{\phi}_k^T = \sum_{k=1}^{N} \frac{A_k^T}{-i\omega - \lambda_k} + \frac{\bar{A}_k^T}{-i\omega - \bar{\lambda}_k}
\end{align*}
\]

where \( \phi_k, \lambda_k \) are respectively the mode shape and pole of mode \( k \) and \( A_k = d_k \phi_k \phi_k^T \) is residue matrix with \( d_k \) constant. The relation between \( \lambda_k \), natural angular frequency \( \omega_k \) and damping ratio \( \xi_k \) is: \( \lambda_k = -\xi_k \omega_k + i\omega_k \sqrt{1-\xi_k^2} = -\xi_k \omega_k + i\bar{\omega}_k \). The symbols \((.)\) and \((.)^T\) denote respectively conjugate and transpose operators.

For \( \tau \in (\infty, +\infty) \), \( R_{xx}(\tau) \) and \( \hat{R}_{xx}(\omega) \) are obtained by

\[
\begin{align*}
R_{xx}(\tau) &= R_{xx}^+(\tau) H(\tau) + R_{xx}^-(\tau) H(-\tau) \\
\hat{R}_{xx}(\omega) &= \hat{R}_{xx}^+(\omega) + \hat{R}_{xx}^-(\omega)
\end{align*}
\]

where \( H(\tau) \) represents the Heaviside step function.

2.2 Continuous wavelet transform

2.2.1 Definition

The continuous wavelet transform of a signal \( x(t) \) can be seen as the projection of the signal onto a base function generated from an analyzing function \( \psi(.) \) called mother wavelet by dilatation and translation [18, 19, 20]:

\[
T_\psi[x](b,a) = \langle x(t), \psi_{b,a}(t) \rangle = \frac{1}{a^p} \int_{-\infty}^{+\infty} x(t) \psi \left( \frac{t - b}{a} \right) dt,
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product whereas the two parameters, \( b \) and \( a > 0 \) are related respectively to time and scale. The shifted and scaled copies of \( \psi \) being denoted \( \psi_{b,a} \) when the p-norm is chosen, is

\[
\psi_{b,a}(t) = \frac{1}{a^p} \psi \left( \frac{t - b}{a} \right)
\]
where $p$ is real strictly positive: $\|\psi_{b,a}\|_p = \|\psi\|_p$

The choice of $p = 1$ was used by Carmona et al. [19] while Mallat [20] preferred $p = 2$. Other normalizations may be found in the literature, and can be more convenient depending on the physical phenomenon under study [19].

The CWT in Equation (3) can also be evaluated in the frequency domain using Parseval-Plancherel’s relation as

$$
T_\psi[x](b, a) = \frac{1}{a^{\frac{p}{2}}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{x}(\omega) a \hat{\psi}(a\omega) e^{i\omega\tau} d\omega,
$$

(5)

where the hat symbol indicates the Fourier transform of a function.

### 2.2.2 Continuous wavelet transform of the power spectral density matrix

The application of the CWT with the $p$-norm to the power spectral density matrix is defined as

$$
T_\psi[\hat{R}_{xx}](b, a) = \frac{1}{a^{\frac{p}{2}}} \int_{-\infty}^{+\infty} \hat{R}_{xx}(\omega) \hat{\psi}(a\omega) d\omega,
$$

(6)

where $a > 0$ and $b$ are now relative to scale and frequency respectively. When the mother wavelet $\psi(\omega)$ is regressive i.e. its Fourier transform $\hat{\psi}(\tau)$ vanishes for $\tau \geq 0$, the following relation is verified [14]

$$
T_\psi[\hat{R}_{xx}](b, a) = \frac{1}{a^{\frac{p}{2}}} \int_{-\infty}^{+\infty} \hat{R}_{xx}(\omega) \hat{\psi}(a\omega) d\omega = T_\psi[\hat{R}_{xx}^+](b, a)
$$

(7)

Substituting $\hat{R}_{\omega}^+(\omega)$ from Equation (11) into Equation (7) and taking integration in complex domain with the regressive mother Cauchy wavelet [13] and the regressive Morlet mother wavelet [14]

$$
T_\psi[\hat{R}_{xx}](b, a) = 2\pi \sum_{k=1}^{N} \left[ A_k \frac{1}{a^{\frac{p}{2}}} \varphi \left( -i\lambda_k - b \right) + A_k \frac{1}{a^{\frac{p}{2}}} \varphi \left( i\lambda_k - b \right) \right]
$$

(8)

where $\lambda_k = -\xi_k \omega_k + i\tilde{\omega}_k$, $\bar{\lambda}_k = -\xi_k \omega_k - i\tilde{\omega}_k$ and $\varphi(.)$ is the complex conjugate formula of the mother wavelet i.e. $\varphi(\omega) = \overline{\psi}(\omega)$

### 3. Frequency-scale domain decomposition method

Since the mother wavelet is well localized in the time and frequency domains, an appropriate choice of the mother wavelet’s parameters allows to amplify the contribution of one mode by reducing that of other modes. For instance, if mode $k$ is highlighted, in its neighbourhood ($b \geq 0$, $a > 0$), from Equation (8), the following approximation is obtained:

$$
T_\psi[\hat{R}_{xx}](b, a) \approx 2\pi A_k \frac{1}{a^{\frac{p}{2}}} \varphi \left( \frac{(\tilde{\omega}_k - b) + i\xi_k \omega_k}{a} \right)
$$

(9)

and the modulus of $T_\psi[\hat{R}_{xx}](b, a)$ is

$$
|T_\psi[\hat{R}_{xx}](b, a)| \approx 2\pi \frac{1}{a^{\frac{p}{2}}} \varphi \left( \frac{(\tilde{\omega}_k - b) + i\xi_k \omega_k}{a} \right) |A_k|_{f(b,a) > 0}
$$

(10)

where $|A_k|$ is the matrix composed of the absolute values of the components of $A_k$. Once mother wavelet parameters are chosen, in the frequency-scale $(b, a)$ plane, the variation of $|T_\psi[\hat{R}_{xx}](b, a)|$
depends on \( f(b,a) \) and reaches its maximum at point \((b_m, a_m)\). Using singular value decomposition of \( T_\psi[R_{xx}](b,a) \), the first singular value is dominant and

\[
\sigma_1(b,a) \approx cf(b,a)
\]  

(11)

where \( c > 0 \) is a constant and thus \( \sigma_1(b,a) \) reaches also its maximum at the point \((b_m, a_m)\). It was demonstrated in references [13, 14] that with an appropriate choice of the norm \( p \), the coordinates of the point \((b_m, a_m)\) give modal parameters as: \( b_m = \hat{\omega}_k \), \( a_m = \xi_k \omega_k \) and the corresponding first singular vector \( \mathbf{u}_1(b_m, a_m) \) is a mode shape estimate of \( \phi_k \).

### 3.1 Choice of mother wavelet parameters

The choice of mother wavelet parameters should allow the extraction of mode \( k \) from its neighbouring modes, which are characterized by the frequency distance \( \Delta \omega_k = \min(|\omega_{k+1} - \omega_k|, |\omega_k - \omega_{k-1}|) \). It depends thus on the localization properties of the wavelet transform in the frequency and time domains. Table I summarizes the definition parameters of the Cauchy and Morlet mother wavelets.

<table>
<thead>
<tr>
<th>Table 1: Mother wavelet parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>formula ( \psi(\omega) )</td>
</tr>
<tr>
<td>( \frac{1}{(1+i\omega)^{\alpha+i\beta}} )</td>
</tr>
<tr>
<td>( \varphi(\omega) )</td>
</tr>
<tr>
<td>norm ( p )</td>
</tr>
</tbody>
</table>

Upper and lower bounds for the mother wavelet’s parameters are given in Equation (12) where \( \Delta f \) is the frequency step of the power spectral density matrix \( R_{xx}(\omega) \).

\[
\begin{align*}
\alpha_{\text{min}} &= \frac{1}{2} \left[ 1 + \left( \frac{5 \xi_k \omega_k}{2 \Delta \omega_k} \right)^2 \right] \leq \alpha \leq \frac{1}{2} \left[ \frac{-5 + \sqrt{25 + 4 \xi_k \omega_k}}{2 \Delta \omega_k} - 1 \right] = \alpha_{\text{max}} \\
0 < \delta &\leq \frac{5 \xi_k \omega_k}{\beta \Delta \omega_k} = \delta_{\text{max}} \\
\beta_{\text{min}} &= \max \left\{ \frac{5}{8}, \frac{2}{\delta} \right\} \leq \beta \leq \frac{\xi_k \omega_k}{\Delta f} - \frac{5}{\sqrt{2} \delta} = \beta_{\text{max}}
\end{align*}
\]  

(12)

### 3.2 Step-by-step procedure of the FSDD method

The FSDD method is performed using three main steps as following:

- **Step 1**: Compute \( T_\psi[R_{xx}](b,a) \). Depending on the chosen mother wavelet, the choice of its parameters enabling a mode to be isolated, for instance mode \( k \), is based on the proposition given in Equation (12).
- **Step 2**: Apply the singular value decomposition to \( T_\psi[R_{xx}](b,a) \) to obtain singular values \( \sigma_1(b,a) > \sigma_2(b,a) \ldots \sigma_S(b,a) \) and their corresponding singular vectors \( \mathbf{u}_1(b,a), \mathbf{u}_2(b,a) \ldots \mathbf{u}_S(b,a) \) where \( S \) is the number of sensors or the size of the square matrix \( R_{xx}(\omega) \).
- **Step 3**: Determine modal parameters using the first singular value \( \sigma_1(b,a) \) and the first singular vector \( \mathbf{u}_1(b,a) \).

In the neighborhood of mode \( k \) on the frequency-scale plane, the maximum value for \( \sigma_1(b,a) \) is first identified, and has coordinates \((b_m, a_m)\). The modal parameters are then deduced as:

- angular frequency: \( \omega_k = \sqrt{\frac{b^2_m + a^2_m}{a_m^2}} \)
- damping ratio: \( \xi_k = \frac{\omega_m}{\sqrt{b^2_m + a^2_m}} \)
- mode shape: \( \phi_k \) is proportional to \( \mathbf{u}_1(b_m, a_m) \)
4. Validity tests

4.1 Numerical example

A mass-spring-damper system of two-Degrees-of-Freedom (DoF) was used for numerical validation. It is presented in Figure 1 with mechanical properties. The exact values of modal parameters were determined and they are given in Table 2.

Using the exact modal parameters of the system, the power spectral density matrix \( \hat{R}_{xx}(\omega) \) based on Equations (1) and (2), was built with \( d_1 = 1.25 \) and \( d_2 = 5 \). The PSD curves were sampled with 4096 points and the frequency step \( \Delta f = 0.0037 \) Hz. The FSDD method described in section 3.2, was applied to the PSD curves. The mother wavelet parameters were taken with \( \alpha = 2 \) and \( \delta = 1 \), \( \beta = 10 \) for the Cauchy and Morlet mother wavelets respectively. The first singular value in the frequency-scale of the Morlet mother wavelet based FSDD method, is shown in Figure 2. Two stationary points corresponding to two modes are indicated by ’+’ symbol. The coordinates of the maxima were used to estimate frequencies and damping ratios, whereas the corresponding first singular vectors were mode shape estimates. Identified results of the FSDD method for both mother wavelets, are compared to exact values in Table 2. An excellent agreement between exact values and identified ones can be noted.

![Figure 1: Two-degrees-of-freedom system](image)

Table 2: 2DoF system, identified modal parameters

<table>
<thead>
<tr>
<th>Modal parameter</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSDD</td>
<td>Exact</td>
</tr>
<tr>
<td>Morlet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cauchy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>3.55</td>
<td>3.55</td>
</tr>
<tr>
<td>Damping (%)</td>
<td>2.26</td>
<td>2.28</td>
</tr>
<tr>
<td>Mode shape</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

4.2 Experimental test

Responses of a cantilever beam were used to check the validity of the FSDD method. The beam of Dural material is of 850 mm in length and a rectangular cross-section of 40 mm \( \times \) 4.5 mm. Time responses were recorded by three PCB Piezotronic accelerometers at 150 mm, 500 mm and 830 mm respectively from the clamped end while white noise excitation was generated by a LSD 201 shaker connected at 700 mm. Equipments of the experimental test are shown in Figure 3. The FSDD method
Figure 2: 2DoF system, $\log(\sigma_1(b, a))$ and stationary points $(b_m, a_m)$ identified by '+’ marker

Figure 3: Laboratory test: instrumented beam

with the Morlet and Cauchy mother wavelets, was applied to the beam’s responses. Identified modal parameters are presented in Table 3. They are perfectly similar and very close to the results given by the time-frequency domain decomposition (TFDD) method [21].

Table 3: Laboratory test: identified modal parameters from the FSDD method

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Morlet</td>
<td>Cauchy</td>
<td>Morlet</td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>19.80</td>
<td>19.86</td>
<td>63.41</td>
</tr>
<tr>
<td>Damping ratio(%)</td>
<td>0.49</td>
<td>0.47</td>
<td>0.45</td>
</tr>
<tr>
<td>Mode shape</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>-2.23</td>
<td>-2.22</td>
<td>-1.54</td>
</tr>
</tbody>
</table>
5. Conclusions

The frequency-scale domain decomposition method, based on two popular mother wavelets of Morlet and Cauchy, has been proposed to identify modal parameters from ambient vibration responses. The analytical foundation of the method is first presented. This allows then to propose a practical step-by-step procedure. The validity of the FSDD method is finally verified with a numerical example and a laboratory experimental test. Whatever the mother wavelet used (Morlet or Cauchy), identified results of modal parameters by the FSDD method are very close to exact values or identified results given by another established modal identification method. In comparison to other wavelet-based methods, the proposed FSDD method enables identification of modal parameters without ridge extraction step.

REFERENCES


