INTERACTION OF A MOVING MECHANICAL OSCILLATOR WITH A PERIODICALLY SUPPORTED INFINITE STRING

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The coupled dynamics of an infinite string with heavy axial tension supported by periodic discrete supports and a three degrees of freedom mechanical oscillator moving at a uniform subcritical speed is studied using substructure synthesis method. String displacements are found by applying a well known model available in the literature, whereas, the steady state response of the oscillator is determined through computing the receptance or dynamic stiffness associated with the continuum. Results show that, near the critical speed, the coupled dynamics is quite different from a prior study by the same authors (where the string was supported by a homogeneous viscoelastic layer) due to the discrete nature of the support condition. It is concluded that this type of behaviour is owing to excessive wave scattering from each support point. The problem analysed in this present work has a great application in pantograph-catenary coupled system as well as in railway tracks.

Keywords: Coupled dynamics, Wave propagation, Infinite string, Mechanical oscillator, Periodic support.

1. Introduction

With the increase in high speed transportation system, researches on infinite continua have gained a significant attention for past few decades [1]. It has a direct application on problems like railway track dynamics, pantograph-catenary system, bridges, ropeway cables etc. [2, 3, 4, 5]. Analyses on beams and taut strings are, therefore, extensively done in this regard. Results obtained using many simple models on these continua have shown excellent agreement with practical situations. In literature, many researchers have expressed their interest on the response of the continua under moving load. Dieterman and Metrikine [6] found out steady state displacement of a beam supported by viscoelastic layer under moving load. Gravrilov [7] worked with similar type of problem, but using string as the continuum, with accelerated load and studied passage of the load through sonic speed. Metrikine [8] carried out research on string under non-linear viscoelastic layer by means of phase plane method. A remarkable analysis was performed by Metrikine and Bosch [9] on the overhead equipment of railway by modelling it as a periodically supported two-level infinite string system with equidistantly spaced discrete elements between the two strings. Several authors have proposed different methodologies in solving the moving load problem. Fourier transformation [9], Green’s function method [10] are some of them.

A substantial amount of research has also been carried out on mechanical systems moving against infinite continua. Kruse et al. [11] reported that some of the eigen frequencies of the mechanical system can be uniquely determined based on the system parameter. It has also been noticed by some
of the researchers that the mechanical system may experience instability if it moves at a speed equal to or greater than the phase velocity (termed as critical speed of moving load or oscillator) of travelling waves in the continuum due to the presence of anomalous Doppler waves [12, 13, 14]. Roy et al. [15], therefore, excluded this possibility in their study by restricting the speed of oscillators below the critical speed limit. They performed a detail analysis on mutual interaction between several oscillators mediated through the continuum while moving at a uniform speed.

Interaction of mechanical oscillator with periodically supported continuum is not well documented in the literature according to the authors’ knowledge. However, the problem has a great application in practical life, e.g., pantograph-catenary coupled system as well as in railway track dynamics. In this present work, therefore, the coupled dynamics between an infinite taut string periodically supported by discrete elements and a three degrees of freedom (DOF) mechanical oscillator moving at a constant speed has been studied. The oscillator has been provided with a base vibration of certain frequency. The method of substructure synthesis [15] is used to perform the analysis. The periodically supported string and the oscillator are considered as two different substructures. First the response of the infinite string has been found out by replacing the oscillator with a harmonic point force moving at the same speed using a well established methodology developed by Metrikine and Bosch [9]. The receptance or dynamic stiffness associated with the string is, then, computed. Finally, the response amplitude of the oscillator has been determined using the obtained dynamic stiffness. This work can be seen as an extension of prior research by the same authors [15].

2. Mathematical model

An infinite string of linear density $\mu$ with constant axial tension $T$ is supported by uniformly spaced discrete mass-spring-damper elements as shown in Fig. 1. The properties of the discrete supports are $m$, $k$, and $c$, respectively, and let, the spacing between the elements be $l$. Consider a mechanical oscillator (Fig. 2) which is moving at a constant speed of $v$ along the axial direction of the string keeping a firm contact with it. The speed of the oscillator is assumed to be well below the phase velocity ($c_p$) of the travelling wave in the continuum. The base of the oscillator is provided with harmonic excitation of frequency $\Omega$.

$$f(t) = F e^{i\Omega t} \delta(x- vt)$$

Figure 1: Infinite taut string supported by discrete elements (substructure 1).

The coupled dynamics of this complex system has been solved using the method of substructure synthesis. Therefore, the response of the infinite string is first found out using Fourier transform method [9].

2.1 Model of the periodically supported string [9]

The partial differential equation (PDE) of the string is given by

$$\mu w_{tt} - Tw_{xx} = Fe^{i\Omega t} \delta(x- vt) \quad (1)$$
where \( w(x,t) \) is the transverse displacement field function of the string, \( F \) and \( \Omega \) are the magnitude and frequency of the moving point force, respectively. The notation \( \delta(\cdot) \) denotes the Dirac delta function with \( x \) and \( t \) being the spatial and temporal coordinates, respectively. \( [\cdot]_{\cdot}^{(\cdot)} \) represents partial derivative of a quantity with respect to the variable (or variables) written in the subscript. Equation (1) needs to be satisfied at each point where the string is attached to the discrete elements, which leads to the following boundary conditions:

\[
 w(nl + 0, t) = w(nl - 0, t) \tag{2}
\]

\[
 T[w_x(nl + 0, t) - w_x(nl - 0, t)] = kw(nl, t) + cw_{tt}(nl, t) + mw_{ttt}(nl, t) \tag{3}
\]

where \( n \) is an integer which represents a span. Using Eqs. (2) and (3), one can get two equations in four unknowns at each boundary. To have a unique solution to the problem, the periodicity condition [9] is further introduced, which is given by

\[
 w(x + nl, t) = w(x, t)e^{i\Omega nl/v}. \tag{4}
\]

The steady state solution of the problem can be analytically obtained using Fourier transformation which takes the equations to the frequency domain. As a consequence, the PDE effectively becomes an ordinary differential equation (ODE) in spatial coordinate. Equations (1) - (4), thus, take the following forms:

\[
 \tilde{w}'' + (\omega^2/c_p^2)\tilde{w} = -(F/Tv)e^{i(\omega+\Omega)x/v} \tag{5}
\]

\[
 \tilde{w}(nl + 0, \omega) = \tilde{w}(nl - 0, \omega) \tag{6}
\]

\[
 T[\tilde{w}'(nl + 0, \omega) - \tilde{w}'(nl - 0, \omega)] = (k - i\omega c - m\omega^2)\tilde{w}(nl, \omega) \tag{7}
\]

\[
 \tilde{w}(x + nl, \omega) = \tilde{w}(x, \omega)e^{i(\omega+\Omega)nl/v} \tag{8}
\]

where \( (.)' \) represents derivative with respect to the spatial coordinate, \( c_p = \sqrt{T/\mu} \) is the phase velocity of travelling waves in the continuum and \( \tilde{w}(x, \omega) \) is defined as

\[
 \tilde{w}(x, \omega) = \int_{-\infty}^{\infty} w(x,t)e^{i\omega t} dt. \tag{9}
\]
The response of any \( n^{th} \) span is, therefore, obtained from Eq. (5) as

\[
\tilde{w}_n(x, \omega) = \left( A_n e^{i\alpha x/c_p} + B_n e^{-i\alpha x/c_p} \right) + Ce^{i(\omega+\Omega)x/v}, \quad (n-1)l \leq x \leq nl
\]  

where \( A_n, B_n \) are arbitrary constants and \( C = Fv c_p^2 / T [c_p^2 (\omega + \Omega)^2 - \omega^2 v^2] \). The solution is obtained in two parts - the complementary function (associated with the homogeneous part of the equation and written within the parentheses) and the particular integral (related to the inhomogeneous part). The first part of the solution is due the free vibration of the string while the second one represents the steady state solution to the harmonic load. Imposing the boundary conditions characterised by Eqs. (6) and (7) along with the periodicity condition (Eq. (8)), one can uniquely determine the unknown coefficients and write the response in terms of the coefficients obtained for the first span. Therefore, Eq. (10) can be recast as

\[
\tilde{w}_n(x, \omega) = \left( A_1 e^{i\alpha x/c_p} + B_1 e^{-i\alpha x/c_p} \right) e^{i(\omega+\Omega)(n-1)l/v} + Ce^{i(\omega+\Omega)(x+(n-1)l)/v}, \quad 0 \leq x \leq l.
\]  

The response in time domain can easily be calculated by performing inverse Fourier transformation of Eq. (11), which is defined as \( w(x,t) = (1/2\pi) \int_{-\infty}^{\infty} \tilde{w}(x, \omega) \exp(-i\omega t) d\omega \). The integration has been carried out numerically to obtain the response of the string which needs to be evaluated at \( x = vt \) for further calculations described in subsequent sections.

### 2.2 Determination of receptance

In order to use the substructure synthesis method, it is required to calculate the receptance or dynamic stiffness of the string associated with the point of application of the moving load. For a single harmonic load the receptance is given by

\[
\alpha(\Omega) = \frac{w(vt,t)}{F v c_p} \quad (12)
\]

where \( w(vt,t) \) is calculated as described in Section. 2.1. The dynamic stiffness which is defined as \( \alpha(\Omega)^{-1} \) can easily be obtained using Eq. (12). Dynamic stiffness is used to find out the contact force exerted on the mechanical system due to the interaction of it with the string.

### 2.3 Response of the mechanical oscillator

The continuum is in continuous contact with the mechanical oscillator which moves at a uniform speed. The speed of the oscillator is purposefully kept in the subcritical speed regime \((v < c_p)\) to avoid possible instability in the oscillator [12]. The base of the dynamical system is excited by a displacement excitation of \( y_0 = \text{Re}[Y_0 \exp(i\Omega t)] \). The equations of motion are, therefore, given by

\[
m_1 \dot{y}_1 + k_1 y_1 + c_1 \dot{y}_1 - k_2 y_2 - c_1 \dot{y}_2 = -f(t)
\]

\[
m_2 \dot{y}_2 + (k_1 + k_2) y_2 + (c_1 + c_2) \dot{y}_2 - k_1 y_1 - c_1 \dot{y}_1 - k_2 y_3 - c_2 \dot{y}_3 = 0
\]

\[
m_3 \dot{y}_3 + (k_2 + k_3) y_3 + (c_2 + c_3) \dot{y}_3 - k_2 y_2 - c_2 \dot{y}_2 = k_3 y_0 + c_3 \dot{y}_0
\]

where \( m_i, k_i, c_i \) are the system parameters (for \( i = 1, 2, 3 \)) of the mechanical system and \( f(t) \) is the contact force coming from the string to the topmost mass of the oscillator. During steady state we can easily write

\[
f(t) = \text{Re}[F \exp(i\Omega t)]
\]

and it is further assumed

\[
y_i = \text{Re}[\tilde{y}_i \exp(i\Omega t)]
\]

where \( \tilde{y}_i \)'s are the complex amplitudes of the masses of the oscillator. Since the mechanical system maintains a firm contact with the string, \( y_1 \) is the same as the displacement of the continuum at \( x = vt \),
i.e., $y_1 = w(\nu t, t)$. Moreover, the force at the same point is given by Eq. (16). This helps us to immediately conclude

$$\ddot{y}_1 = \alpha(\Omega)F.$$  \hspace{1cm} (18)

Replacing Eqs. (16) and (17) into Eqs. (13) - (15) and using the relation characterised by Eq. (18), the following matrix equation is obtained

$$\mathbf{P}\tilde{\mathbf{Y}} = \mathbf{Q}$$  \hspace{1cm} (19)

where $\tilde{\mathbf{Y}} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^\top$, $\mathbf{P}$ is a matrix of system parameters with elements defined as

$$
\begin{align*}
P_{11} &= \{-m_1\Omega^2 + k_1 + i\omega c_1\} + \alpha(\Omega)^{-1} \\
P_{12} &= P_{21} = -(k_1 + i\omega c_1) \\
P_{13} &= P_{31} = 0 \\
P_{22} &= \{-m_2\Omega^2 + (k_1 + k_2) + i\omega(c_1 + c_2)\} \\
P_{23} &= P_{32} = -(k_2 + i\omega c_2) \\
P_{33} &= \{-m_3\Omega^2 + (k_2 + k_3) + i\omega(c_2 + c_3)\}
\end{align*}
$$

and $\mathbf{Q} = (k_3 + i\omega c_3)(0, 0, Y_0)^\top$. Hence, the response amplitude vector $\tilde{\mathbf{Y}}$ is determined as

$$\tilde{\mathbf{Y}} = \mathbf{P}^{-1}\mathbf{Q}. \hspace{1cm} (20)$$

Similarly, the contact force on the topmost mass of the oscillator can be calculated using Eq. (18).

### 3. Results and discussion

In this section the interaction of a mechanical system with an infinite taut string periodically supported by discrete mass-spring-damper elements is discussed. Table 1 shows the parameters used for the taut string while Table 2 represents the data for the mechanical system. The dynamics of the string under moving harmonic load is studied first, and then, the responses of the mechanical system while interacting with the continuum are presented. No speed restriction has been kept during the analysis of the string under moving point load. Figure 3 shows the responses of the continuum at different speeds and frequencies of the moving load. It is seen that at subcritical speeds two waves of different wave numbers are generated at the point of application of the load. The wave propagating upstream is of higher wave number than that of the wave travelling downstream. At supercritical
The steady state response of the topmost mass of a mechanical dynamic system moving against the continuum is, then, determined when subjected to a base excitation of unit amplitude. Since the response is a complex function of frequency and speed, the non-dimensional steady state response
of the topmost mass is shown as a three dimensional plot as well as a contour plot in Fig. 5. In the contour plot (Fig. 5(b)) the reddish yellow zones depict the peaks of the response surface while the blue zones indicate the valleys. At lower frequencies (0-2 Hz) the response surface is similar to that of [15]. However, in moderate and higher frequencies the nature of the surface is completely different. A number of scattered peaks is observed at these regions unlike the peaks appeared in [15] along a particular frequency-line in speed-frequency plane. Moreover, the density of peaks is more near the critical speed value. As discussed earlier, periodic discrete supports are mostly responsible for that.

4. Conclusions

An infinite string supported by discrete mechanical elements and subjected to a moving dynamic oscillator with base excitation has been studied using substructure synthesis method. Dynamics of the string when acted upon by a moving harmonic point load is found out using Fourier transformation method as described in [9]. It is found that the discrete nature of the supports becomes evident at speeds close to the critical value. Otherwise at low frequencies the discrete supports can be modelled as a viscoelastic layer. Displacement of the string at the loading point near the critical speed is quite different compared to that of the continuum supported by viscoelastic layer. The response of the oscillator is, then, determined by computing the dynamic stiffness associated with the string. The steady state condition has showed that, close to critical speed, the dynamics of the oscillator is always different from what was obtained using a continuous viscoelastic support in [15]. When the speed of the oscillator approaches the critical speed value, the wave length of the travelling waves generated by the disturbance of the oscillator becomes sufficiently small compared to the spacing of the discrete supports. This increases the phenomenon of wave scattering from each support points, resulting in higher values of steady state response in the oscillator.

REFERENCES


