MODAL AND FINITE ANALYSIS OF A ROTATING PIEZOELECTRIC BEAM

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In this paper, the modal and finite analysis of a rotating piezoelectric beam are studied by considering electromechanical coupling. The differential equations governing two transverse vibrations are derived by the extended Hamilton principle, wherein an additional piezoelectric coupling term is present in two transverse planes in contrast to the traditional models of rotating beam. The ordinary differential equations are investigated by using Galerkin method. The nature frequency and complex mode are obtained by solving the standard eigenvalue problems of the gyroscopic system. It shows that the rotating speed makes the frequency of system split into two parts. The forward and backward nature frequency and complex mode of the rotating piezoelectric beam are discussed. For natural vibration of a rotating piezoelectric beam, the gyroscopic effect is studied during modal motions in this study. Further, the finite analysis of a rotating beam is investigated to compare with the results of numerical simulation. The modal motions described the forward and backward processions.

Keywords: rotating piezoelectric beam, finite analysis

1. Introduction

The rotating piezoelectric beam consists of a rectangle beam and four piezoelectric materials in the two transverse planes, as shown in Fig. 1. Recently, there has been a growing appetite of research workers to study piezoelectric material on energy harvester [1, 2] and micro-structure [3] for rotating system. The free vibrations features of rotating piezoelectric beam should be studied.

As an effective tool to study the responses and mode interactions, the modal analysis of gyroscopic systems is very important. But, when dealing with gyroscopic continuum, the modal analysis becomes difficult because the complex modes [4] should be considered. Uspensky [5] investigated the normal mode under a forced excitation by utilizing the invariant manifold method. To research the free vibrations of a gyroscopic system, Arvin and Nejad [6] described the complex dynamical characteristics of nonlinear normal modes. Recently, there are lots of valuable papers towards normal modes of undamped [7-9] and damped systems [10, 11] by using numerical simulation. Pan et al. [12] utilized complex modal technique to calculate the frequency and complex mode of serpentine belt drives.


In this paper, the governing differential equations of piezoelectric beam are obtained and the nature frequency as well as complex mode are discussed. The modal analysis is considered for the gyroscopic system and the results are validated by finite element method. The forward and backward nature frequencies of the piezoelectric rotating beam are investigated.
2. Governing equations of a rotating piezoelectric gyroscope

Fig. 1 shows the structure of a vibrating piezoelectric beam. The length \((L_1, L_2, L)\) and width \((w_b, w_p)\) of beam and piezoelectric materials are shown in the figure. For all the parameters used in this study, subscripts \(b\) denotes the beam and \(p\) denotes piezoelectric material. The \(x-y-z\) coordinate system represents inertial frame and with a rotating speed \(\Omega\) around \(x\) axis.

Figure 1: A rotating beam with surrounded four piezoelectric films.

The kinetic energy of a rotating Rayleigh beam can be obtained by using two Euler angles transformation [15]

\[
T = \frac{1}{2} \int_0^L \left[ m \left( \dot{w}^2 + \dot{v}^2 + \dot{u}^2 \right) + m\Omega^2 \left( w'^2 + v'^2 \right) + 2m\Omega \left(v\dot{w} - w\dot{v} \right) \right] ds +
\]

\[
\frac{1}{2} \int_0^L \left[ j \left( \dot{v}^2 + \dot{w}^2 \right) + 2j\Omega^2 \left( v'^2 + w'^2 \right) \right] ds + 2JL\Omega^2
\]

where

\[
m = w_b\rho_b h_b + 4w_p (H_{L_1} - H_{L_2}) \rho_p h_p, \quad j = \int_A \rho(s) z^2 dA = \int_A \rho(s) y^2 dA, \quad \rho(s) = \rho_b + (H_{L_1} - H_{L_2}) \rho_p, \quad H_{L_1} = H(s - L_1), \quad i = 1, 2.
\]

In (2), \(H(s)\) is Heaviside function, \(\rho_{b,p}\) and \(h_{b,p}\) are the mass density and thickness, respectively. \(A\) is the cross section of beam, \(m\) is the total mass, \(\rho(s)\) is total density and \(j\) is total inertia of beam and piezoelectric material.

The mechanical characteristic of piezoelectric material is coupled with its electric characteristics. For the configuration considered here, the electrical displacement is one dimensional and the stress strain relation for the piezoelectric material is known as follows [16, 17]

\[
T_p = c_{11}^p S - e_{31} S + e_{33}^s E_3, \quad D = e_{31} S + e_{33}^s E_3, \quad S = -\rho_2 + \varepsilon_{33} E_3, \quad E_3 = \frac{V_{v,w}}{h}
\]

where \(c_{11}^p\) is the stiffness coefficient, \(e_{33}^s\) is the dielectric permittivity, \(T_p\) is the stress, \(S\) is the strain, \(e_{31}\) is the piezoelectric strain constant, \(D\) is the electrical displacement, \(E_3\) is the electrical field and \(V_{v,w}\) is the voltage. The relations between voltage and current are

\[
V_v = Z\dot{Q}_v, \quad V_c = Z\dot{Q}_c
\]

where \(Z\) is electrical resistance, \(Q\) is charge quantity.

The total potential energy of rotating piezoelectric beam can be derived...
\[ U = \frac{EI}{2} \int_0^L \left( \rho_2^2 + \rho_3^2 \right) ds + \frac{\theta}{2} \int_0^L \left( (-\rho_2)Z\dot{Q}_w + \rho_3\dot{Q}_r \right) ds \]
\[ -\frac{C_p}{2} \int_0^L \left( (Z\dot{Q}_w)^2 + (Z\dot{Q}_r)^2 \right) ds \]  

(5)

where \( EI = c_{11}^b I_p + (H_{t_i} - H_{t_s}) c_{11}^p I_p \) is bending stiffness, \( C_p = w_p e_{33}^s L(H_{t_i} - H_{t_s})/h_p \) is piezoelectric film capacitor, \( \theta = w_p e_{31} (h_i/2 + h_s) (H_{t_i} - H_{t_s}) \) is electromechanical coupling coefficient, and

\[ I_p = \frac{1}{12} w_p h_3^3, \quad I_p = 2[\frac{1}{12} w_p h_3^3 + w_p h_3 \left( \frac{h_i}{2} + \frac{h_s}{2} \right) ^2]. \]

(6)

The virtual work done electrical resistance is

\[ \delta W = -\frac{1}{2} \int_0^L (c_v \delta v + c_w \delta w + Z\dot{Q}_w \delta Q_w + Z\dot{Q}_r \delta Q_r) ds. \]

(7)

Substituting the results of kinetic, potential energy and virtual work into the Hamilton principle:

\[ \int_t^0 (\delta T - \delta U + \delta W) dt = 0 \]

(8)

and using voltage-current relation, we can obtain the following four partial governing equations

\[ EIv'''' + m(\ddot{v} - \dot{\Omega}w - 2\Omega\dot{w} - \Omega^2 v) - j(\dot{\Omega}w' + \dot{\Omega}v) + c\dot{v} - V_v(t) \theta'' = 0 \]
\[ EIw'''' + m(\ddot{w} + \dot{\Omega}v + 2\Omega\dot{v} - \Omega^2 w) - j(\dot{\Omega}v' + \dot{\Omega}w') + c\dot{w} - V_w(t) \theta'' = 0 \]
\[ C_p \dot{V}_v(t) + \frac{V_v(t)}{Z_v} + \theta \dot{V}_v'' = 0 \]
\[ C_p \dot{V}_w(t) + \frac{V_w(t)}{Z_w} + \theta \dot{V}_w'' = 0 \]

(9)

Substituting periodic voltages \( V_v = \bar{V}_v e^{i\omega t} \) and \( V_w = \bar{V}_w e^{i\omega t} \) into Eq. (9), the last two equations of (9) will be eliminated by using displacements to replace voltages, and then the equations can be reduced to two DOF partial differential equations

\[ m(\ddot{v} - \dot{\Omega}w - 2\Omega\dot{w} - \Omega^2 v) + 2j\Omega^2 v'' - j\dot{v}'' + cv + EIv''' + \frac{\theta^2 v''''}{C_p (1 + Z_v/Z)} = 0, \]
\[ m(\ddot{w} + \dot{\Omega}v + 2\Omega\dot{v} - \Omega^2 w) + 2j\Omega^2 w'' - j\dot{w}'' + cw + EIw''' + \frac{\theta^2 w''''}{C_p (1 + Z_w/Z)} = 0 \]

(10)

where \( Z_v = 1 / i \omega C_p \).

We consider the free-supported boundary conditions \( v''=w''=0 \) and \( v''''=w''''=0 \) at both ends, a Galerkin method is used

\[ v(s,t) = \sum_{r=1}^M \phi_r(s) q_r(t), \]
\[ w(s,t) = \sum_{r=1}^N \phi_r(s) p_r(t) \]

(11)

and

\[ \phi_r(s) = \sin \alpha_r s + \sinh \alpha_r s - \sinh \alpha_r L - \sin \alpha_r L \left( \cosh \alpha_r s + \cos \alpha_r s \right), \]
\[ \cos \alpha_r L \cdot \cosh \alpha_r L = 1 \]

(12)

where \( r \) is mode number, \( p \) and \( q \) are generalized temporal coordinates.
Substituting Eq. (11) into Eq. (10), assuming a constant base angular velocity $\Omega$ and neglecting damping, one can obtain two ordinary differential equations

$$
\sum_{j=1}^{M} \left( \int_{0}^{L} m(s) \phi_j(s) ds \right) \ddot{q}_j - \sum_{j=1}^{M} \left( \int_{0}^{L} \phi_j(s) ds \right) \dot{q}_j - 2\Omega^2 \sum_{j=1}^{N} \left( \int_{0}^{L} m(s) \phi_j(s) ds \right) \dot{\phi}_j
+ EI \sum_{j=1}^{M} \left( \int_{0}^{L} \phi_j''(s) ds \right) q_j - \Omega^2 \sum_{j=1}^{M} \left( \int_{0}^{L} m(s) \phi_j(s) ds \right) \dot{q}_j + 2 j\Omega^2 \sum_{j=1}^{N} \left( \int_{0}^{L} \phi_j'(s) ds \right) q_j
$$

\[
\theta^2 \sum_{j=1}^{M} \left( \int_{0}^{L} \phi_j''(s) ds \right) q_j
+ \frac{\theta^2}{C_p L^2 (1 + Z_0 / Z)} = 0, \quad i = 1, 2, ..., M,
\]

\[
\sum_{j=1}^{N} \left( \int_{0}^{L} m(s) \phi_j(s) ds \right) \ddot{\phi}_j - \sum_{j=1}^{N} \left( \int_{0}^{L} \phi_j(s) ds \right) \dot{\phi}_j + 2\Omega^2 \sum_{j=1}^{M} \left( \int_{0}^{L} m(s) \phi_j(s) ds \right) \dot{q}_j
+ EI \sum_{j=1}^{N} \left( \int_{0}^{L} \phi_j''(s) ds \right) p_j - \Omega^2 \sum_{j=1}^{N} \left( \int_{0}^{L} m(s) \phi_j(s) ds \right) \dot{p}_j + 2 j\Omega^2 \sum_{j=1}^{N} \left( \int_{0}^{L} \phi_j'(s) ds \right) p_j
\]

\[
\theta^2 \sum_{j=1}^{N} \left( \int_{0}^{L} \phi_j''(s) ds \right) p_j
+ \frac{\theta^2}{C_p L^2 (1 + Z_0 / Z)} = 0, \quad i = 1, 2, ..., N.
\]

3. Analysis of the nature frequency

In this section, the free vibration of Eq. (14) is studied by numerical calculation. Substituting $q(s,t)=C\phi(s)e^{i\omega t}$ and $p(s,t)=C\phi(s)e^{i\omega t}$ into Eq. (14), and using the mode number $r=2$, we can obtain the first two order frequencies. By using these parameters

$$
L = 50 \times 10^{-3} m; \rho_s = 7800 kg / m^3; \rho_p = 7900 kg / m^3; E_b = 210 \times 10^9 N / m^2;
$$

$$
E_p = 63 \times 10^9 N / m^2; \omega_w = 4.1 \times 10^{-3} m; h_y = 4.14 \times 10^{-3} m; \varepsilon_{33} = 0.283 \times 10^{-4} F / m,
$$

$$
L_2 - L_1 = 22.9 \times 10^{-3} m, d_{31} = -274 \times 10^{-12} C / N; \omega_p = 1.9 \times 10^{-3} m; h_p = 0.5 \times 10^{-3} m.
$$

Figure 2: The first two nature frequencies vs. rotating speed.

In Fig. 2, the first two nature frequencies versus rotating speeds are plotted. The small value is the first order frequency, and the big value is the second order frequency. Every order frequency split into two parts with the rotating speeds. For each value of rotating speed, there are two nature frequencies corresponding for the two gyroscopic modes of the piezoelectric beam. One forward frequency is increasing and the other one backward frequency is decreasing. We will use the four typical points
$A(\omega_1=8769 \text{ Hz}), B(\omega_1=8788 \text{ Hz}), C(\omega_2=23960 \text{ Hz})$ and $D(\omega_2=23980 \text{ Hz})$ on Fig. 2 in the analysis of modal motions in the next section.

4. Modal analysis by using numerical and finite method

In this section, the modal motions has been studied in Fig. 3 by using the same parameters in Eq. (15) based on the extracting eigenvectors method.

The snapshots for different rotating cases ($\Omega=0$ and $\Omega=60$) in a period of the rotating piezoelectric beam for the first two orders modal motions are shown in Fig. 3. When the rotating speed is zero, the modal shape of first and second systems are shown in the Fig 3a, b. The modal shape is same with a static beam. The modal motions are corresponding to the four typical points $A$, $B$, $C$ and $D$ in Fig. 2c-f. The first-order modal motions exhibited in Fig 3c, d. From the figures we can find Fig 3c is backward whirling but Fig 3d is forward whirling. There are two nodes in the first-order modal shape. The second-order modal motions exhibited in Fig 3e, f. From the figures we can find Fig 3e is backward whirling but Fig 3f is forward whirling. There are three nodes in the second-order modal shape. The elliptical trajectory shows the gyroscopic effect.
Figure 3: The complex mode functions derived by the gyroscopic systems.

In Fig. 4, the modal and frequency are discussed by using finite element method. Figs. 4a, b are the first-order modal shape when the rotating speed $\Omega=60$ rad/s. The frequencies are 8647.9 Hz and 8787.5 Hz, the numerical results are 8769 Hz and 8788 Hz. It is close with each other. Figs. 4c, d are the second-order modal shape when the rotating speed $\Omega=60$ rad/s. The frequencies are 22255 Hz and 22517 Hz, the numerical result is 23960 Hz and 23980 Hz. The reason for the difference is that the numerical calculation is only two orders, so the first order result is accurate. If we use more orders to deal with the dispersed systems, we can obtain more accurate frequencies. From the Fig. 4 we can also found the modal shape is same with the analysis in the Fig. 3.

![Figure 3: The complex mode functions derived by the gyroscopic systems.](image1)

**Figure 4:** The complex mode functions derived by the finite analysis.

5. **Conclusion**

The free vibrations of a rotating piezoelectric beam are investigated by both finite and numerical methods. The governing equations are derived by using Hamilton principle. The frequencies and complex modes are obtained by solving the standard eigenvalue problems of the linearized system, and then validated by finite element method. Some conclusions are shown as follows.

1. In the free vibration analysis of a rotating piezoelectric beam, forward frequency and backward frequency are considered at a free-supported boundary conditions.
2. The whirling motions of the gyroscopic complex modes have been illustrated. The forward procession and backward procession of the gyroscopic continua have been discussed by using the numerical calculation.
3. The gyroscopic effect has been confirmed by modal analysis method and finite element method.

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