This paper used the gauge invariance approach to treat acoustic multiple scattering. The symmetry used is time reversal symmetry. This scattering mechanism is that in the process of time reversal or backpropagation, each inhomogeneity point will become a scattering point and provides information on the medium. Hence the more the degree of inhomogeneity the more information it can provide and thus increase the image resolution.

Keywords: multiple scattering, gauge invariance approach

1. Theory of Time-Reversal Acoustics

Time reversal acoustics is based on the principle of time-reversal invariance of the acoustic wave equation in lossless media[1–4]. It means that, if \( \varphi(\vec{r}, t) \) is an acoustic field and thus is a solution of the wave equation, then \( \varphi(\vec{r}, -t) \) is another solution and thus a possible acoustic field. In particular, if \( \varphi(\vec{r}, t) \) is a wave diverging from a point, \( \varphi(\vec{r}, -t) \) must be focused on this same location. This gives rise to the novel idea of focusing procedure in two steps: step one, an acoustic source is installed and its emitted field is measured by means of a closed receiving surface around the medium. We suppose that each point of this surface is able to record the wave as a function of time and then to re-emit in step two in order to generate the time-reversal solution. This time-reversed wave is backpropagated through the medium and finally focuses on the locations of the initial source.

However, the concept of closed time-reversed cavity (CTRC) is difficult to realize[3] and is usually replaced by a time-reversed medium (TRM) of finite size which shows of performance comparable to the cavity in spite of the loss of information[4]. We will consider the case of solid-fluid interface which is applicable to situations of nondestructive testing and medical imaging.

C Draeger et al [5] consider a point like source of elastic waves located inside a solid half space at the origin \( x = y = z = 0 \) [Figure 1(a)]. The plane solid-fluid interface is at \( z = h > 0 \) and the TRM is located in the fluid at \( z = Z > H \). The source emits a short pulse of longitudinal and transverse waves (or P and S waves) which are partially terminated at the interface into the fluid and thus both converted into pressure waves. They consider that all waves emitted or reflected into the negative z directions are lost. SH waves in particular, are totally reflected and therefore it is impossible to apply the time-reversed
process to this polarization of transverse waves. This results in a limitation of the TRM device. That is why the P and SV wave components yield two wave fronts in the fluid.

The TRM records them both and is able to distinguish between the two wave fronts if their interval time is sufficiently distinct. In this case, they can choose to backpropagate only the wave fronts corresponding to the P waves or the SV waves or both. When the backpropagating wave fronts arrive at the fluid-solid interface, each of them creates two wave fronts in the solid, one corresponding to the original type of waves, thus wanted and one of the other type of wave, thus unwanted [Fig 1(b)]. Also the unwanted SV wave, created by the returned P wave front and the unwanted P wave due to the returned SV wave front yield a low-level noise and they are not focused and arrive at different times. On the contrary, the two wanted wave fronts focus at the same place, that is, the location of the initial source which now remain passive.

![Time Reversal Mirror](image)

**Fig.1** Time-reversal behaves as a two-step process. (a): Emission of a short pulse of longitudinal and transverse waves by an active source in the solid, yielding two pressure – wavefronts in the fluid; Recording by the TRM; (b): Re-emission of the time-reversed fields into the fluid by the TRM; Backpropagation of the two wavefronts in the fluid, yielding four wavefronts in the solid; The desired waves arrive simultaneously at the initial source location, the undesired ones arrive before and afterwards.(From C. Draeger, D. Cassereau, and M.Fink[1997]).

Their derivations here are for the P and SV waves propagating in the positive z direction.[5] They describe the displacement vector \( \vec{u} \) of the incident elastic field as a function of the scalar and vector potentials \( \phi \) and \( \psi \) of the P and SV waves propagation in the positive \( z \) direction: \[ \vec{u} = \nabla \phi(x, y, z, t) + \nabla x \nabla \times (0, 0, \psi(x, y, z, t)) \] (1)

The initial field can be for a point like source or extended source. The only condition is the existence of the 2D Fourier transform over \( x \) and \( y \) in the frequency domain of the potential \( \Phi(k_x, k_y, z, \omega) \) and \( \Psi(k_x, k_y, z, \omega) \): \( \phi \) is the scalar potential and is given rise to the longitudinal-wave field.
\[ \phi(x, y, z, t) = \left( \frac{1}{2\pi} \right) \int d\omega \, \phi(x, y, z, \omega) \exp(-j\omega t) \quad (2) \]

and

\[ \tilde{\phi}(x, y, z, \omega) = \left( \frac{1}{2\pi} \right)^2 \iint dk_x dk_y \Phi(k_x, k_y, z, \omega) \exp\left(j(k_x x + k_y y)\right) \text{FT}^{-1}_{2D}[\Phi] \quad (3) \]

Referring to the acoustic wave equation (1), the dependence on \( z \) can be written explicitly as

\[ \Phi(k_x, k_y, z, \omega) = \Phi(k_x, k_y, z = 0, \omega) \exp(jv_a(k_r)z) \quad (4) \]

where \( v_a \) is a function of \( k_r = \sqrt{k_x^2 + k_y^2} \) and the longitudinal wave speed \( \alpha \) and is defined by

\[ \sqrt{\omega^2/\left(\alpha^2 - k_r^2\right)}, \quad \text{if} \quad k_r \leq \frac{\omega}{\alpha} \]

\[ -j\sqrt{(k_r^2 - \omega^2)/\alpha^2}, \quad \text{if} \quad k_r > \frac{\omega}{\alpha} \]

If \( v_a \) is real, i.e. if \( k_r \leq \frac{\omega}{\alpha} \) it can be considered as the \( z \) component \( k_z \) of the wave vector \( k = (k_x, k_y, k_z) \) of a propagative wave.

If \( v_a \) is imaginary the wave is called evanescent wave and is non-propagative. Its amplitude decreases exponentially with the depth in the \( z \) direction.

\( \Psi \) is the vector potential and it gives rise to the shear wave.

Following same procedure as for the longitudinal wave one obtains:

\[ \Psi(k_x, k_y, z, \omega) = \Psi(k_x, k_y, z = 0, \omega) \exp(jv_\beta(k_r)z) \quad (6) \]

Each wave type in the solid generates at the interface a wave front in the fluid. Describing the transmitted sound wave by its pressure field \( P \), they can split it formally into the past created by the P wave \( P_p \) and the one from the SV wave \( P_s \):

\[ P(x, y, z, t) = P_p(x, y, z, t) + P_s(x, y, z, t) \quad (7) \]

where

\[ P_p(k_x, k_y, z, \omega) = \Phi(k_x, k_y, z = 0, \omega) T_{ps} \exp(jv_a h) \exp(jv_c(z - h)) \quad (8) \]

and
\[ P_s(k_x, k_y, z, w) = \Psi(k_x, k_y, z = 0, \omega) T_{sf} \exp(jv_\beta h) \exp(jv_c(z - h)) \]  
\text{and } T_{pf}, T_{sf} = \text{transmission coefficients.} 

The derivation of the transmission coefficients are as follows: 

The incident plane sound waves are on a solid-fluid or fluid-solid interface. For simplicity, the interface is at \( z = 0 \) and \( k_y = 0; k_x = k_r \). Three cases are considered: (1) incident P wave, (2) incident SV wave. Referring to equation (1), the expression of the trial wave fields are

Case (1)

\[
\begin{align*}
\tilde{\phi} &= \exp(j(k_xx + v_\alpha z)) + R_{pp}\exp(j(k_xx - v_\alpha z)) \\
\tilde{\psi} &= R_{ps}\exp\left(j(k_xx - v_\beta z)\right) \\
P &= T_{pf}\exp\left(j(k_xx + v_c z)\right)
\end{align*}
\]

where \( R_{pp}, R_{ps} = \text{reflection coefficients} \)

Case (2)

\[
\begin{align*}
\psi &= \exp\left(j(k_xx + v_\beta z)\right) + R_{ss}\exp\left(j(k_xx - v_\beta z)\right) \\
\phi &= R_{sp}\exp(j(k_xx - v_\alpha z)) \\
P &= T_{sf}\exp\left(j(k_xx + v_c z)\right)
\end{align*}
\]

The fields are related to each other by boundary conditions at the interface \( z = 0 \) \[^8,^9\]:

(1) The z component of the displacement \( u_z \) is continuous

\[
\tilde{u}_z = \frac{\partial}{\partial z} \tilde{\phi} - \frac{\partial^2}{\partial x^2} \tilde{\psi} = 1/\omega^2 \rho \frac{\partial}{\partial z} \tilde{p}
\]

(2) The vertical traction is continuous \[^7\] :

\[
T_z = \lambda \tilde{\nu}^2 \tilde{\phi} + 2\mu \frac{\partial^2}{\partial z^2} \tilde{\phi} - 2\mu \left( \frac{\partial^3}{\partial x^2 \partial z} \right) \tilde{\psi} = -p
\]

where \( \lambda \) and \( \mu \) are the Lamé coefficients and related to the wave speeds by \( \lambda + 2\mu = \rho_s \alpha^2 \) and \( \mu = \rho_s \beta^2 \).

(3) The horizontal traction \( T_x \) vanishes\[^5\] .
\[ T_x = 2\mu \frac{\partial^2}{\partial x \partial z} \tilde{\phi} - \mu \frac{\partial}{\partial x} \left( \frac{\partial^2}{\partial x^2} \tilde{\psi} - \frac{\partial^2}{\partial z^2} \tilde{\psi} \right) = 0 \]

Eliminating the reflection coefficients for each case described above, the transmission coefficients are obtained as:

\[ T_{pf} = \frac{2}{N} \rho_s \beta^4 (\omega^2 - 2\beta^2 k_r^2) v_\alpha \]

\[ T_{ss} = \frac{4(-jk_r)}{N} \rho_s \beta^2 \omega^4 k_r v_\alpha v_\beta \]

\[ T_{fp} = \frac{2}{N} (\omega^2 - 2\beta^2 k_r^2) v_c \]

\[ T_{fs} = \frac{-4}{N(-jk_r)} \beta^2 k_r v_\alpha v_c \]

where \( N = 4 \rho_s \beta^4 k_r^4 v_c + 4 \rho_s \beta^4 k_r^2 v_\alpha v_\beta - 4 \rho_s \beta^2 \omega^2 k_r^2 v_c + \rho_s \omega^4 v_c - \rho \omega^4 v_\alpha \).

At \( z = 2 \), the TRM records the arriving field. Here they consider the simpler case of phase conjugation in the frequency domain. They assume that the mirror is infinitely large, i.e. it measures and emits at each point in the whole \( xy \) plane at \( z = Z \). They omit the aperture function to simplify the forward mathematics. From equations (8) and (9) They obtain for the time reversed wave fronts:

\[ \tilde{P}_{p}^R(k_x, k_y, z, \omega) = \Phi^*(k_x, k_y, z = 0, \omega) T_{pf}^* \exp(-jv_\alpha h) \exp(-jv_c^*(z - h)) \exp(-jv_c(z - Z)) \]

and

\[ \tilde{P}_{s}^R(k_x, k_y, z, \omega) = \Psi^*(k_x, k_y, z = 0, \omega) T_{sf}^* \exp(-jv_\beta h) \exp(-jv_c^*(z - h)) \exp(-jv_c(z - Z)) \]

Each of them generates two wave fronts in the solid. Hence they obtain two desired wave fields corresponding to the initial type of wave,

\[ \tilde{\Phi}_{p}^R(k_x, k_y, z, \omega) = \Phi^*(k_x, k_y, z = 0, \omega) \exp(-jv_\alpha z) T_{pf}^* T_{fp} \exp(-2lm v_\alpha h) \exp(-2lm v_c (z - h)) \]

and

\[ \tilde{\Phi}_{s}^R(k_x, k_y, z, \omega) = \Psi^*(k_x, k_y, z = 0, \omega) \exp(-jv_\beta z) T_{sf}^* T_{fs} \exp(-2lm v_\beta h) \exp(-2lm v_c (z - h)) \]
and two undesired wave fronts, one SV wave created by a returned P wave and vice versa:

$$\tilde{\Phi}^R_s(k_x, k_y, z, \omega) = \tilde{\Psi}^*(k_x, k_y, z = 0, \omega) \exp(-j\nu_\alpha z) T^*_{sf} T^*_{fp} \exp(j\nu_\alpha) \exp(-2lmv_c(z - h))$$  \hspace{1cm} (17)

and

$$\Phi^R_p(k_x, k_y, z, \omega) = \tilde{\Phi}^*(k_x, k_y, z = 0, \omega) \exp(-j\nu_\beta z) T^*_{pf} T^*_{fs} \exp(j(\nu_\beta - \nu_\alpha) h) \exp(-2lmv_c(z - h))$$  \hspace{1cm} (18)

With $\nu_\alpha = \nu_\alpha^* + 2jIm\nu_\alpha$ and $FT_{2D}[\Phi^*(\exp(-j\nu_\alpha^*))] = \tilde{\phi}^*(x, y, z, \omega)$, the returned and desired P wave of equation (15) can be written in the $xy$ space as a convolution:

$$\tilde{\phi}^R_p(x, y, z, \omega) = \left(\frac{1}{2\pi}\right)^2 \tilde{\phi}^*(x, y, z, \omega) FT_{2D}[T^*_{pf} T^*_{fp}] \exp(-2lmv_\alpha(h - z)) \exp(-2lmv_c(z - h))$$  \hspace{1cm} (19)

The first term $\tilde{\phi}^*(x, y, z, \omega)$ corresponds exactly to the time-reversed P field $\phi(x, y, z, -t)$ we are interested in. But the quality of the reversed wave decreases by losses due to interface and propagation. In the same procedure, the returned, desired SV wave can be written as

$$\tilde{\psi}^R_s(x, y, z, \omega) = \left(\frac{1}{2\pi}\right)^2 \tilde{\psi}^*(x, y, z, \omega) FT_{2D}[T^*_{sf} T^*_{fs}] \exp(-2lmv_\beta(h - z)) \exp(-2lmv_c(z - h))$$  \hspace{1cm} (20)

$\psi^*(x, y, z, \omega)$ corresponds to the time-reversed SV field $\psi(x, y, z, -t)$. This means in particular that, if the TRM reverses both fields together, they will focus at the same time at the same place, i.e. the initial source location. This is the proof of the time reversal mirror capability of spatial and temporal recompression.

The undesired wave fronts do not arrive at the same time as the desired ones at the origin. The time-reversed wave front corresponding to the P wave generates at the interface two wave fronts in the solid. A desired one of P wave type and an undesired one of SV wave type which propagates more slowly than the first one and hence accumulate later at the initial source position in the same way, the time-reversed wave front correspond to the SV wave generates an undesired P wave front in the solid which arrives later at the initial source positions. In the same way, the time-reversed wave front corresponds to the SV wave generates an undesired P wavefront in the solid which arrives sooner than the desired one. So they find the main propery of the undesired wave. They arrive at the wrong time, they are not focused and they are of low amplitude. The TRM can reverse the P and SV waves but cannot reverse the SH polarization. The reversed P and SV waves arrive at the same time, focused on the initial source location with a focal spot width corresponding approximately to their central wavelength. Thus the slower transverse waves are better focused.

2. **Time reversal acoustics and superresolution**

In time-reversal acoustics, a signal is recorded by an arraytransducers, time-reversed and then propagates back through the medium and refocuses approximately on the source that emitted it. The refocusing is approximate because of the finite size of the aperture of the array of transducers (receivers and transmitters) which is only a certain portion of the 3D time reversal cavity. It is often small compared...
to the propagation distance, so only a small part of the advancing wave is captured and time reversed. In a homogeneous medium, the refocusing resolution of the time-reversed signal is limited by diffraction. However when the medium has random inhomogeneities, the refocusing effect is better and the resolution of the refocused signal can in some circumstances beat the diffraction limit. This is superresolution. In homogeneous media, the spatial resolution of the time-reversed signals is limited by diffraction and is inversely proportional to the aperture size and proportional to the wavelength times the propagation distance. Time reversed signals propagate backwards through the time-independent medium and go through all the multiple scattering, reflection and refraction that they underwent in the forward direction. That is why refocusing occurs.

If the medium is randomly inhomogeneous, the focusing equation of the backpropagated signal can be better than the resolution in the homogeneous case. This is referred to as super-resolution. The random inhomogeneities produce multipathing and the TRM appears to have an aperture that is larger than its physical size, an effective aperture $a_e > a$. This means that the recompressed pulse is narrower than in the homogeneous medium and we have super-resolution with a spatial scale of order $\lambda L/ a_e$; where $L$ is distance from the source to the TRM and $a$ is the size of the TRM. This phenomenon was observed in underwater acoustics experiments (Dowling and Jackson 1990[6]; Hodgkiss et al, 1999[7]; Kuperman et al 1997[8]) as well as in the ultrasound regime (Derode et al 1995[9]; Fink, 1997[10], 1999[11]).

Practical examples of this illustration of superresolution have been demonstrated with computer simulation by Peter Blomgren, George Papanicolaou and Hongkai Zhao [12] for underwater acoustical imaging. They presented a detailed analytical and numerical study of how multipathing in random media enhances resolution in time-reversal acoustics. That is how superresolution arises in random media. They have shown that when the propagation distance is large compared to the wavelength and the correlation length of the inhomogeneities and the time-reversal minor is small, there is an exact expression for the effective size of the TRM, its effective aperture valid in both in the time and frequency domain. Multipathing makes the effective size of the TRM much larger than its physical size.

Sean Klehman and Anthony Devaney[13] have demonstrated superresolution applied to seismic imaging. They have applied their theory to the case where the transmitter and receiver sensor arrays need not be coincident and for cases where the background medium can be nonreciprocal. Their theory is based on the singular value decomposition of the generalized multistatic data matrix of the sensor system rather than the standard eigenvector/eigenvalue decomposition of the time-reversed matrix as was employed in other treatments of time-reversal imaging. They derived a generalized multiple signal classification (MUSIC) algorithm that allows superresolution imaging of both well-resolved and nonwell-resolved point targets from arbitrary sensor array geometry. Their time-reversal MUSIC algorithm is tested and validated in two computer simulations of offset vertical seismic profiling where the sensor sources are aligned along the earth’s surface and the receiver array is aligned along a subsurface borehole. Their results demonstrated the high contrast, high resolution imaging capability of their new algorithm combination when compared with classical backpropagation or field focusing.

3. Time reversal acoustics is a form of gauge invariance

The above shows the application of time reversal acoustics to scattering by inhomogeneities which is a form of multiple scattering. Time reversal acoustics is a form of gauge invariance as the replacement of time $t$ by $-t$ in the acoustic field equation resulting in no change in the form of the equation.
REFERENCES