THEORETICAL AND EXPERIMENTAL STUDIES OF $H_\infty$ CONTROL FOR VEHICLE SUSPENSION SYSTEMS WITH CONTROL DELAY

Peilin Li, Mingxia Fang
School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai, China
email: 92226@tongji.edu.cn

The vehicle suspension system considering time delay is taken as the research object, and the combination methods of theory and experiment is used to study the influence of time delay on the vibration characteristics of the vehicle suspension system. Firstly, the dynamic model of active suspension system considering time delay is established. A matrix inequality for system asymptotic stability is derived by using the Lyapunov-Krasovskii functional and free-weighting matrix method. Based on the matrix inequality, a $H_\infty$ controller is designed with known maximum time delay of the system. Then, numerical simulation and experimental methods are used to verify the time-delay control strategy. The results show that the control law in this paper can effectively suppress the sprung mass acceleration. Finally, the relationship between the time delay and the RMS value of the sprung mass acceleration under different gains is analyzed. It’s found that better control effect could be obtained when the delay $d$ and $\tau$ are close.

Keywords: active suspension; time-delay; $H_\infty$ control

1. Introduction

It is widely acknowledged that an active suspension system is the effective way to improve suspension performance, compared to a passive suspension. Many active suspension control approaches are presented based on various control strategies such as linear quadratic Gaussian control, adaptive control [1, 2], fuzzy logic control [3, 4], and $H_\infty$ control [5-7]. It’s been widely accepted that the $H_\infty$ control method is an applicable strategy for active suspension systems, especially in the context of robustness and disturbance attenuation [8].

However, in the active control process of the car, the time delay is unavoidable due to factors such as signal acquisition, transmission, controller calculation and actuator actuation. The study found that even a small time delay has a great impact on system stability, which could make the system unstable and divergent [9]. But a reasonable time delay could also be benefit [10, 11]. In the past decades, the stability and control problem about time delay systems have been widely investigated [12-14].

In general, some results have been obtained on the time delay system. However, the experimental research on the time delay system is relatively few, because the small time delay is difficult to measure. In this paper, the two-degree-of-freedom suspension system considering time-delay is taken as the research object. The delay control strategy based on $H_\infty$ robust control is designed by Lyapunov-Krasovskii functional, and the time-delay control results are verified by experimental methods.
2. Problem statement

Consider a quarter car active suspension system with control time delay[14] as shown in Fig. 1, where spring \( k_s \) and damper \( c_s \) consist of the passive suspension; \( k_t \) and \( c_t \) stand for the tire stiffness and tire damper respectively; \( u(t-d(t)) \) is the active force generated by an actuator considering the time delay \( d \), which satisfies \( 0 < d(t) \leq \tau \) and \( \dot{d}(t) \leq \mu \); \( m_s \) and \( m_u \) represent sprung and unsprung masses, respectively. The vertical displacements of the sprung and unsprung masses with respect to their static positions are denoted by \( z_s \) and \( z_u \) respectively while the vertical road disturbance is denoted by \( z_r \).

\[ \begin{align*}
\ddot{z}_s + c_s (z_s - z_u) + k_s (z_s - z_u) &= u(t-d(t)) \\
\ddot{z}_u + c_s (z_u - z_s) + k_s (z_u - z_s) + k_t (z_u - z_r) + c_s (z_u - z_r) &= -u(t-d(t))
\end{align*} \] (1)

Automotive active suspension control is a typical multi-target control. The main objective is to design a control strategy for the active suspension system, so that the closed-loop controlled suspension system can meet the following conditions. Firstly, the ride comfort should be improved. The ride comfort generally quantified by the sprung acceleration, so we choose sprung acceleration as the performance control output denoted by \( w_1 \). In order to improve the vehicle ride comfort, the main goal is to minimize the \( H\infty \) norm of the transfer function from the disturbance \( \xi(t) = \dot{z}_r \) to the control output \( w_1 = \ddot{z}_s \).

Besides, the suspension stroke should not exceed the allowable maximum, that is \( |z_s(t) - z_u(t)| \leq z_{\text{max}} \), and \( z_{\text{max}} \) is the maximum suspension deflection. And in order to ensure uninterrupted contact between the wheels of the vehicle and the road surface, the dynamic tire load should not exceed the static tire load \( k_t (z_u(t) - z_r(t)) < (m_s + m_u)g \). As a result, we choose the suspension stroke \( z_s(t) - z_u(t) \), relative dynamic tire load \( k_t (z_u - z_r) / (m_s + m_u)g \) as constrained control outputs denoted by \( w_2 \). In addition, the actuator output force cannot exceed its limit \( |u(t)| \leq u_{\text{max}} \), where \( u_{\text{max}} \) is the maximum possible actuator control force.

Defining the state vector \( Z(t) = [z_s - z_u \ z_u - z_r \ \dot{z}_s \ \dot{z}_u]^T \), the state-space equations of the active vehicle suspension system can be given as:

\[ \begin{align*}
\dot{Z} &= A(t)Z(t) + B(t)u(t-d(t)) + D(t)\xi(t) \\
\dot{w}_1 &= C_1 Z(t) + D_1 u(t-d(t)) \\
\dot{w}_2 &= C_2 Z(t)
\end{align*} \] (2)

where:
\[
A(t) = \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
- \frac{k_s}{m_s} & 0 & - \frac{c_s}{m_s} & \frac{c_s}{m_s} \\
- \frac{k_s}{m_u} & - \frac{k_i}{m_u} & - \frac{c_s}{m_u} & - \frac{c_s}{m_u} \\
\end{bmatrix}, \quad B(t) = \begin{bmatrix}
0 \\
0 \\
\frac{1}{m_s} \\
- \frac{1}{m_u} \\
\end{bmatrix}, \quad D(t) = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\]

\[
C_1 = \left[ - \frac{k_s}{m_s} \ 0 \ - \frac{c_s}{m_s} \ - \frac{c_s}{m_s} \right], \quad D_1 = \frac{1}{m_s}, \quad C_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & - \frac{k_i}{(m_s+m_u)g} & 0 & 0 \\
\end{bmatrix}.
\]

Introducing the state feedback controller \( u(t) = KZ(t - d(t)) \) for the system, the Eq. (2) is transformed into:

\[
\dot{Z} = AZ(t) + B_k Z(t - d(t)) + D_k \zeta(t) \\
w_1 = C_1 Z(t) + D_1 Z(t - d(t)) \\
w_2 = C_2 Z(t)
\]

where \( B_k = BK, D_k = D_1 K \).

For the above performance requirements, the control problem of the active suspension can be described as follows: design an internal stable controller to minimize the \( H_\infty \) norm of the control system Eq. (3) from the interference \( \zeta(t) \) to the output \( w_1(t) \); and suppress \( w_2 \) and \( u \) in the time domain to guarantee \( |\{w_2(t)\}_q| \leq \{w_{2,\max}\}_q, q = 1, 2, t > 0 \) and \( |u(t)| \leq u_{\max} \), where \( w_{2,\max} = [\zeta_{\text{max}} \ 1]^T \).

3. State feedback \( H_\infty \) control strategy

In this section, we use the state feedback \( H_\infty \) control strategy to solve the control problem of the active suspension system with time delay.

3.1 \( H_\infty \) control strategy for known upper limit of time delay

Theorem 1: Considering the closed-loop system in Eq. (3). For given scalars \( \tau > 0, \mu, A, B_k, D, C_1, C_2 \) and \( D_k \), if there exist matrices \( P > 0, Q > 0, S > 0, R > 0, N_1 \) and \( N_2 \) with appropriate dimensions such that the following LMIs hold:

\[
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & PD & \tau A^T & \tau N_1 & C_i^T \\
* & \Xi_{22} & 0 & \tau B_k^T & \tau N_2 & D_k^T \\
* & * & -\gamma^2 I & \tau D^T & 0 & 0 \\
* & * & * & -\tau R^{-1} & 0 & 0 \\
* & * & * & * & -\tau R & 0 \\
* & * & * & * & * & -I \\
\end{bmatrix} < 0
\]

where \( \Xi_{11} = PA + A^TP + Q + S + N_1 + N_1^T, \Xi_{12} = PB_k - N_1 + N_2^T, \Xi_{22} = -(1 - \mu)S - Q - N_2 - N_2^T \).
Then we conclude: 1) the closed-loop system is asymptotically stable for the delay \( d(t) \) satisfying \( 0 \leq d(t) \leq \tau \) and \( 0 \leq d(t) \leq \mu \); 2) under zero initial condition, the \( H\infty \) norm of the transfer function from the disturbance \( \xi(t) \) to the control output \( w_1 \) has \( H_{\infty} < \gamma \).

**Proof:** Considering the Lyapunov-Krasovskii functional as follows:

\[
V = Z^T P Z + \int_{t-\tau}^{t} Z^T(s) Q Z(s) ds + \int_{t-d(t)}^{t} Z^T(s) S Z(s) ds + \int_{t-\tau}^{t} \int_{t+\theta}^{t} \hat{Z}^T(s) R Z(s) ds d\theta \geq 0 \tag{5}
\]

where \( P, Q, S \) and \( R \) being symmetric positive definite matrix.

The derivative of \( V(t) \) can be obtained as:

\[
\dot{V} = Z^T (A^T P + PA) Z + Z^T (t - d(t)) B_k^T P Z(t) + Z^T (t) P B_k Z(t - d(t)) + w^T D^T P Z + Z^T P D w + Z^T (Q + S) Z - Z^T (t - \tau) Q Z(t - \tau) - (1 - \dot{d}(t)) S Z(t - d(t)) + \tau \hat{Z}^T(t) R \hat{Z}(t) - \int_{t-\tau}^{t} \hat{Z}^T(t) R \hat{Z}(t) ds
\]

(6)

For any appropriately dimensional matrices \( N_1 \) and \( N_2 \), the following equality holds:

\[
\left[ Z^T(t) N_1 + Z^T(t - d(t)) N_2 \right] \times \left[ Z(t) - Z(t - d(t)) - \int_{t-d(t)}^{t} \hat{Z}(s) ds \right] = 0 \tag{7}
\]

Adding Eq. (7) into the right-hand side of Eq. (6), the following inequality can be obtained after some calculations:

\[
\dot{V} + w^T_1(t) w_1(t) - \gamma^2 \hat{\xi}^T(t) \hat{\xi}(t) \leq \eta^T_2(t) \left( \Sigma + \Psi^T R \Gamma_1 + \Gamma_2^T \Gamma_2 \right) \eta_2(t) - \int_{t-d(t)}^{t} \eta^T_3(t,s) \Psi \eta_3(t,s) ds \tag{8}
\]

where

\[
\Sigma = \begin{pmatrix}
\Xi_{11} + \tau X_{11} & \Xi_{12} + \tau X_{12} & PD \\
\ast & \Xi_{22} + \tau X_{22} & 0 \\
\ast & \ast & -\gamma^2 I
\end{pmatrix},
\Psi = \begin{pmatrix}
X_{11} & X_{12} & N_1 \\
\ast & X_{22} & N_2 \\
\ast & \ast & R
\end{pmatrix},
\eta_2(t) = \left[ \eta_2^T(t) \right],
\eta_3(t,s) = \left[ \eta_3^T(t,s) \right],
\eta_1(t) = \left[ Z^T(t) \right],
\eta_2(t) = \left[ Z^T(t - d(t)) \right],
\eta_3(t,s) = \left[ Z^T(t - d(t)) \xi^T(t) \right],
\eta_1(t) = [A \ B_k \ D], \ \Gamma_1 = [C_1 \ D_k], \ \Gamma_2 = [C_2 \ D_k].
\]

Using the theorem about matrix inequality, we can see that the following inequalities are satisfied on the conditions of Eq. (4)

\[
\Sigma + \Psi^T R \Gamma_1 + \Gamma_2^T \Gamma_2 < 0, \ \Psi > 0 \tag{9}
\]

and then we have

\[
\dot{V} + w^T_1(t) w_1(t) - \gamma^2 \hat{\xi}^T(t) \hat{\xi}(t) < 0 \tag{10}
\]

Assuming the interference input is zero, we have \( \dot{V}(x(t)) < 0 \) which means the system is asymptotically stable. Under zero initial conditions, we have \( V(0) = 0 \) and \( V(+) \geq 0 \). Integrating both sides of Eq. (10) yields \( \|w_1(t)\|_2 < \gamma \|\hat{\xi}(t)\|_2 \) for \( \forall \xi(t) \in L_2[0, \infty) \). The proof is completed.

In addition, the suspension system needs to meet output constraints and maximum control constraints. According to[14], the following inequalities should be satisfied.
\[
\begin{bmatrix}
\{w_{2,\text{max}}^{2}\}_q P & \sqrt{\rho} (C_q^2)^T \\
\ast & -I
\end{bmatrix} < 0, q = 1, 2
\] (11)

\[
\begin{bmatrix}
-u_{\text{max}}^2 P & \sqrt{\rho} K^T \\
\ast & -I
\end{bmatrix} < 0
\] (12)

3.2 Processing of nonlinear matrix inequality

Since Eq. (4) is a nonlinear matrix inequality, it is difficult to solve directly. Therefore, using the parameter adjustment method, a new matrix inequality equivalent to the nonlinear matrix inequality is obtained.

Defining new matrices \( W = \begin{bmatrix} P & 0 \\ N_1^T & N_2^T \end{bmatrix} \), \( W^{-1} = \begin{bmatrix} P^{-1} & 0 \\ M_1 & M_2 \end{bmatrix} \) and \( \chi = \text{diag}(W^{-1}, I, I, R^{-1}, I) \), we can get \( N_1^TP^{-1} + N_2^TM_1 = 0 \) and \( N_2^TM_2 = I \) due to \( W \times W^{-1} = \begin{bmatrix} I & 0 \\ N_1^TP^{-1} + N_2^TM_1 & N_2^TM_2 \end{bmatrix} \).

Defining \( M_1 = nP^{-1} \), \( M_2 = mP^{-1} \), \( \bar{P} = P^{-1} \), \( \bar{Q} = P^{-1}QP^{-1} \), \( \bar{R} = R^{-1} \), \( \bar{S} = P^{-1}SP^{-1} \), \( Y = P^{-1}K^T \), where \( n \) and \( m \) are real number, \( m \neq 0 \), inequality (4) can be transformed into the following one through performing congruence transformations to Eq. (4) by \( \chi \).

\[
\begin{bmatrix}
\omega_{11} & \omega_{12} & D & \omega_{14} & 0 & \omega_{16} \\
\ast & \omega_{22} & 0 & \tau mYB^T & \tau \bar{R} & mYD^T \\
\ast & \ast & -\gamma^2 I & \tau D^T & 0 & 0 \\
\ast & \ast & \ast & -\tau \bar{R} & 0 & 0 \\
\ast & \ast & \ast & \ast & -\tau \bar{R} & 0 \\
\ast & \ast & \ast & \ast & \ast & -I
\end{bmatrix} < 0
\] (13)

where

\[
\omega_{11} = \bar{P}A^T + A\bar{P} + nBY^T + nYB^T + \bar{Q} + \bar{S} - n^2 \bar{Q} - n^2 (1 - \mu) \bar{S},
\]

\[
\omega_{12} = (1 - n) \bar{P} + mBY^T - (1 - \mu) nm\bar{S} - nm\bar{Q},
\]

\[
\omega_{22} = -(1 - \mu)m^2 \bar{S} - m^2 \bar{Q} - 2m\bar{P}, \quad \omega_{14} = \tau (\bar{P}A^T + nYB^T), \quad \omega_{16} = \bar{P}C_1^T + nYD_1^T.
\]

Similarly, performing the same congruence transformations to Eq. (11) and (12), the following inequalities holds:

\[
\begin{bmatrix}
-\{w_{2,\text{max}}^{2}\}_q \bar{P} & \sqrt{\rho \bar{P}} (C_2^1)^T \\
\ast & -I
\end{bmatrix} < 0, q = 1, 2
\] (14)

\[
\begin{bmatrix}
-u_{\text{max}}^2 \bar{P} & \sqrt{\rho \bar{Y}} \\
\ast & -I
\end{bmatrix} < 0
\] (15)

After the transformations, the Eq. (13)-(15) are all linear matrix inequality. When the maximum stable time delay of the system \( \tau_{\text{max}} \) is known, for given \( n \) and \( m \), the objective function can be described as:

\[
\min_{n,m} \gamma \quad \text{subject to LMI (13)(14)(15)}
\] (16)
4. Responses analysis of time delay suspension control system

In this section, the numerical method is used to analyze the vibration characteristics of the suspension control system under the above control strategy, and the calculation results are verified on the suspension time-delay control test platform. The parameters of the suspension scale test model are: \( m_s = 136.05 \text{ kg} \), \( m_u = 24.288 \text{ kg} \), \( k_s = 10200 \text{ N/m} \), \( k_t = 98000 \text{ N/m} \), \( c_s = 2100 \text{ N} \cdot \text{s/m} \), \( c_t = 0 \text{ N} \cdot \text{s/m} \). Among them, \( m_s \) and \( m_u \) are obtained by direct weighing, \( k_s \) and \( k_t \) are given by the manufacturer through professional measuring instruments, while \( c_s \) and \( c_t \) are obtained by parameter identification using transfer function method.

In order to obtain the inherent time delay in the control system, it is divided into two parts. The first part includes the time delay caused by the signal acquisition, transmission, processing and control force calculation, which can be directly read by the oscilloscope with a time delay of 37ms. The other part is the time delay caused by the actuator's actuation process. This part refers to the amount of time delay provided by the actuator manufacturer and other's work\cite{15}, taking 28ms. Then the inherent time delay of the entire feedback loop is \( d(t) = 65 \text{ ms} \).

4.1 Numerical simulation

In order to facilitate analysis and comparison, the simulation analysis was carried out for \( d(t) = 65 \text{ ms} \), and the suspension parameters were taken as the suspension scale model parameters. The deterministic function \( \xi(t) = 0.004 \sin(2 \pi ft) \), \( f = 5 \text{ Hz} \) was taken as input excitations. Choosing \( \tau_{\text{max}} = 70, 80 \text{ ms} \) and using the \( H_\infty \) control strategy proposed above, the sprung acceleration responses before and after the control were obtained, as shown in Fig. 2. It can be seen from the Fig. 2(a) that after applying feedback control, the acceleration amplitude of the sprung mass is reduced by 17.07% from 2.2801 m/s\(^2\) to 1.8908 m/s\(^2\), indicating that the active control of the suspension has good damping effect. It can also be seen that the system after control remains stability under the condition of time delay, and no divergence occurs. Similarly, the sprung mass acceleration amplitude is reduced by 16.95% from 2.2801 m/s\(^2\) to 1.8936 m/s\(^2\) when the gain coefficient corresponding to \( \tau_{\text{max}} = 80 \text{ ms} \).

![Figure 2: Sprung mass acceleration calculation results (a) \( \tau_{\text{max}} = 70 \text{ ms} \) (b) \( \tau_{\text{max}} = 80 \text{ ms} \).](image)

4.2 Experimental verification

In order to improve the credibility of the analysis, the calculation results were verified on the 2DOF suspension time-delay control test platform. A suspension control test platform has been constructed with a magnetorheological damper as an actuator, as shown in Fig. 3.
The corresponding matrix gain $K$ when $\tau_{\text{max}}$ = 70 ms, 80 ms are selected for the control experiment test, and the experimental results of the sprung acceleration responses are obtained, as shown in Fig. 4. Comparing Fig. 4 with Fig. 2, it is found that the experimental results agree well with the simulation results. The RMS values of the sprung mass accelerations in experimental results are 1.2231 m/s$^2$ and 1.2635 m/s$^2$, respectively. Compared with the simulation results of the previous section of 1.3370 m/s$^2$ and 1.3388 m/s$^2$, the errors are 8.52% and 5.62%, respectively.

Figure 5 shows the plot of time delay versus sprung acceleration for different control gains. As seen from Fig. 5, the magnitude of the sprung acceleration is related to the choice of $\tau_{\text{max}}$ as well as the size of $d(t)$. For the same curve, the closer $\tau_{\text{max}}$ and $d(t)$ are, the better the control effect.
5. Conclusion

This paper has investigated the time-delay robust $H_\infty$ control problem of Two-DOF suspension system, and experimental verification is carried out. First of all, the dynamic system has been established and the control problem is proposed. Second, by using Lyapunov-Krasovskii functional and free weight matrix method, the time-delay $H_\infty$ controller has been designed to guarantee asymptotic stability of the closed-loop system with $H_\infty$ disturbance attenuation level and satisfy the required output constraints. Then, the above conditions are converted into equivalent linear matrix inequalities according to the parameter adjustment method for convenience of calculation. Finally, the effect of the proposed control strategy is studied by a combination of theory and experiment. Both results show the effectiveness of the $H_\infty$ control strategy.

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