A relationship between the spectrum of a cold jet at 90° and the spectra at arbitrary angles and temperature ratios is explored, by combining limiting forms of analytical solutions of Lilley’s equation for the high and low frequency limits. The model, which extends that presented previously at ICSV13 (2006), considers the sources are as compact. For angles inside the cone of silence, refraction is explicitly included. For the hot jet, it is shown that new components which exist in addition to those contributing to the cold one, can show both dipole and quadrupole efficiency. Predictions are compared with experimental results and it is shown that the model achieves, in general, good to reasonable agreement both for cold and hot jets. For the hot jet, at 30°, however, the model over-predicts the spectra. Empirical corrections are proposed to enhance the prediction capability of the model.

Keywords: jet noise, hot jets, asymptotic models, compact sources

1. Introduction

Although more than 65 years have elapsed since the introduction of the acoustic analogy approach by Lighthill [1], and despite the significant progresses achieved in the prediction of jet noise (e.g., [2, 3], this subject continues to attract intensive research and controversies, notably concerning details of the source structure and mechanisms (for instance, different assumptions or conclusions regarding source coherence are found in [2, 3, 4]). In this paper, a simple point source model developed in [5] and presented at ICSV13, relating the far field noise spectrum of a cold jet at 90° with the spectrum at arbitrary angle and temperature ratio is revisited. The comparison with experiment, previously made for a hot jet with temperature ratio 2.9, in the Mach number range from 0.35 to 0.9, was restricted to 90° only and is now extended to different values of the polar angle θ and to cold jets as well. The model is based in combining high- and low-frequency limits solutions of Lilley’s equation and assumes
compact sources. Refraction is considered, based on [6], by a multiplying factor effective inside the cone of silence. Some parameters are adjusted for best fit, so that the model, although analytically based, ends up being a semi-empirical one. It is expected that the model will help improving the similarity laws given in [6-9].

In sections 2 and 3 the basic theory and assumptions are summarized. Comparison with experimental results of Brown and Bridges [10] is given in section 4, and the conclusions in section 5.

2. The point source model

By using a general representation scheme for the solution of Lilley’s equation, developed in [11, 12] and the particular form of the equation for the pressure fluctuation \( p' \) (discussed in [13]), in which the equivalent source terms in the continuity and momentum equations, \( q \) and \( f - \nabla \cdot T \), are given by

\[
f - \nabla \cdot T = -c_0^2 \nabla \cdot (\rho uu) = \nabla c_0^2 \cdot (\rho uu) - \nabla \cdot (\rho c_0^3 uu) \quad \text{and} \quad q = (\gamma - 1)u \cdot \nabla p' - \nabla \cdot (\rho c_0^2 u'),
\]

where \( p' \) denotes the Fourier transform, \( \gamma \) is the specific heat ratio, Musafir [5] showed that a formal far field solution of the equation can be represented, for \( q = 0 \) (while the first term in \( q \) is possibly negligible [5, 13], the second one will be considered later), by

\[
p'(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(y, \tau) \rho uu + \nabla^2 G \cdot uu \cdot \left( \nabla c_0^2 / c_0^2 + 2C^{-1} \cos \theta \nabla M \right) dV_y d\tau \tag{1}
\]

where \( y \) and \( \tau \) are the source (i.e., emission) variables, \( G \) is the Green’s function of a suitably chosen auxiliary problem, \( M = U/c_\infty \) is the local mach number based on the ambient sound speed \( c_\infty \), \( C = 1 - M \cos \theta \), the \( \nabla \nabla \) operator stands for \( \nabla \nabla \) when all terms that can be expressed as containing an explicit contribution of mean velocity and temperature gradients are removed, so that these contributions appear separately, and \( \nabla^* \) stands for the \( \nabla \) operator without the transverse (for a round jet, radial) component.

Since equation (1) considers explicitly the effects of the mean velocity and temperature gradients, it is acceptable to approximate \( G \) by its ‘plug flow-homogeneous medium’ form, discussed in detail by Dowling et al [14] for the low frequencies; the high-frequency case is addressed by Goldstein [15]. The advantage of considering the ‘plug flow-homogeneous medium’ solution is that, in this case, the needed first and second space derivatives of \( G \) in the far field are obtained, essentially, by multiplying the operator \( (4\pi|x|)^m \left[-c_0^2 \partial^2 / \partial t \partial^m \right] \), where \( m \) is 1 or 2, by directivity factors which account for mean flow effects, and by including the appropriate retarded time. This property permits expressing the Fourier transform of Eq. (1) for a round jet (if the random character of the source is, for the moment, neglected), considering the source space coordinates \( y = (y_1, r, \phi) \), and the fact that \( \rho c_0^2 \approx \rho_0 c_0^2 = \rho_o c_\infty^2 \), as

\[
\bar{p} = \frac{\rho_o}{4\pi|x|} \int \left\{ -k^2 F : \widetilde{uu} + ik \left( g_1 \widetilde{uu} + g_6 \widetilde{uu} \right) \right\} \left( c_0^{-2} \frac{d^2 c_\infty^2}{dr^2} + 2C^{-1} \cos \theta \frac{dM}{dr} \right) dV_y \tag{2}
\]

where \( \sim \) denotes the Fourier transform, \( k = \omega/c_\infty \), \( \omega \) being the angular frequency, the matrix \( F \) contains the (directivity) flow factors \( F_{ij} \) stemming from \( \nabla \nabla G \), which were taken to absorb the factor \( c_\infty^2 C^{-2} \), and \( g_1 \) and \( g_6 \) are those stemming from \( \nabla^* G \), also considered to absorb \( c_\infty^2 C^{-2} \). Unlike [5], the convection factor based on the convection Mach number was not considered here, only the one based on the local Mach number.

By assuming that the sources are compact and that the local gradients can be approximated by typical values, the far field sound power spectral density can be expressed as proportional to
\[
H(\omega) = \frac{k^4D + k^2(g_1^2K_1 + <g_2^2> K_2)}{4C^2 \cos^2 \theta \left( \frac{dM}{dr} \right)^2 + \left( \frac{1}{c_0^2} \frac{dc_0^2}{dr} \right)^2 + 4C^{-1} \cos \theta \frac{dc_0^2}{dr} \frac{dM}{dr}}
\]

where \(H(\omega)\) represents the spectral distribution of the Fourier transform of the double volume integral (over the source coordinate difference, \(\Delta y\) and over \(y\)) of the fourth order two-point correlation \(u_i^2(y,t)u_i^2(y + \Delta y,t + \Delta t)\), self- or cross-correlations involving other components being assumed to differ by a numerical factor; \(D\) stands for the non-dimensional directivity of the terms associated with \(C^2(\nabla \nabla)_h\), being a function of angle \(\theta\), nominal jet Mach number \(M_J\) (angular) frequency \(\omega\) and temperature ratio \(R = c_J^2/c_x^2\), where \(c_J^2\) is proportional to the nominal jet temperature; \(K_1\) and \(K_2\) are coefficients describing the relative importance of the (Fourier transformed) autocorrelations terms associated with \(g_1\) and \(g_2\), and \(< >\) denotes the average over the far field azimuthal angle. The terms proportional to \(k^2\) show quadrupole-like efficiency while those proportional to \(k^4\) present dipole-like efficiency. As will be shown, hypothesis on the gradients of mean properties affect the scaling of some terms.

It is assumed that \(dU/dr = \beta \omega_0\), where \(\beta\) is a coefficient and that \(c_0^2 \frac{dc_0^2}{dr} = 0, d, \) where \(\delta = 2(R-1)/(R+1)\), \(d\) is the jet diameter and \(\epsilon\) is taken as \(\epsilon = (\epsilon_1 \beta k d + \epsilon_2)/M_J\) where \(\epsilon_1\) and \(\epsilon_2\) are, ideally, constants; since it can be shown that the contribution of the neglected \(q = -\nabla \cdot [\rho \mathbf{u}(\mathbf{c}^2)]\) term scales like that of the \(\epsilon_1\) part of the term containing \(c_0^2 \frac{dc_0^2}{dr}\), it was proposed to increase \(\epsilon_1\) but to reduce the \(2\epsilon_1 \epsilon_2\) factor which appears when the full term is squared, to \(\epsilon_1 \epsilon_2 [5]\). With these hypotheses, it was possible to obtain an expression relating the noise spectrum of a cold jet at \(\theta = 90^\circ\), \(S_{\theta_0,cold}\), (which scales as \(k^4\)) with the corresponding spectrum at arbitrary angle and temperature ratio, \(S_{\theta,R}\), as

\[
S_{\theta,R} = S_{\theta_0,cold} D + \left( g_1^2 K_1 + <g_2^2> K_2 \right) \frac{4 \cos^2 \theta \beta^2}{C^2} + \left( \frac{\delta}{M_J} \right)^2 \left( \epsilon_1 \beta^2 + \frac{\epsilon_1 \epsilon_3 \beta}{kd} \right) + \frac{4 \cos \theta \delta}{C} \frac{\epsilon_1 \beta + \epsilon_2}{kd}
\]

(4)

3. High and low frequency limits

In order to model the term which multiplies \(S_{\theta_0,cold}\) in Eq. (4), expressions for the low and high frequency limits of \(D\), \(g_1^2\) and \(<g_2^2>\) were derived, based on the results given in [14] and [15], and combined as

\[
X(kd) = \frac{X_{LF} + a(kd)^n X_{HI}}{1 + a(kd)^n}
\]

(5)

where \(X\) stands for the desired quantity and \(X_{LF}\) and \(X_{HI}\) for their low and high frequency limit expressions; \(a\) and \(n\) are arbitrary constants, to be adjusted for best fit.

It was also assumed, for convenience, after some initial tests, that the uu source is statistically isotropic and also (see discussion in [16]) that \(tr(\mathbf{uu}) = 0\). With these two hypotheses, it can be shown that

\[
D_{LF} = \frac{1}{4R^2} + \frac{1}{C^2} \left[ \frac{9}{4} \cos^4 \theta - \frac{5C^2}{4R} \cos^2 \theta \right] + \frac{3}{(R+C^2)^2} \left( \sin^2 2\theta + \sin^4 \theta \right)
\]

and that \(K_1 = K_2 = 3/4\) (both for the high and low frequencies); the first hypothesis leads to

\[
D_{HI} = \frac{C^2}{R^2}
\]

(7)

The expressions for \(g_1^2\) and \(<g_2^2>\) do not depend on these hypotheses and are given by...
The high-frequency \( D \), however, is valid only outside the cone of silence. Inside (i.e. for \( \cos \theta \geq (M + \sqrt{R})^{-1} \)), a refraction correction applies, being given by the multiplying factor \( [6] \)

\[
e^{-\alpha d \sqrt{\cos \theta - C^2 R^{-1}}} \tag{12}
\]

where \( \alpha \) is the ratio between twice the radial distance travelled by the radiation inside the jet and the jet diameter \( d \). Inside the cone of silence this factor was applied to the full high frequency solution.

### 4. Comparison with experiment

The model was tested against experimental data obtained at NASA Glenn Research Center [10], which consists of sets of 1/3 octave spectra for a \( d = 5.08 \) cm jet, for different Mach number (0.35 \( \leq M_J \) \( < 1.5 \)) and temperature ratios (0.8 \( < R < 2.9 \)). The Mach number at the source was taken as \( M = 0.6M_J \) and, in a somewhat similar way, the temperature ratio \( R \), when appearing explicitly and in the computation of the cone of silence boundary, was replaced by \( R^* \), in which the nominal jet temperature gives place to the average between that temperature and the far field one, yielding \( R^* = (R + 1)/2 \), as discussed in [5]. For obtaining \( \delta \), however, the standard \( R \) was used.

The parameters \( \alpha \) and \( n \) in Eq. (5) where taken, respectively, as 0.25 and 2, while \( \alpha \) was chosen as \( 1/18 (= 0.056) \); \( \beta \) was considered as 0.27 and for cold jets and as 2.7 for the hot jet (\( R = 2.9 \)). All these values were chosen for best fit. For the hot jet, \( \varepsilon_2 \) was taken as 0.8 while \( \varepsilon_2 \) was considered to be Mach number dependent and given by \( \varepsilon_2 = 1.53 - 1.61M_J \) (no longer a constant). These values were also chosen for best fit. Another correction considered was that, for the low frequencies only, in order to reduce mean flow amplification, \( C \) was replaced by \( C^* = (C^2 + \alpha_2^2 M^2 \delta^2)^{0.5} \), with \( \alpha_2 = 0.75 \) for the cold jet and 0.85 for the hot jet.

The results obtained with these constants and expressions are shown in Fig. 1 for the cold jet (\( \theta = 45^\circ \) and \( 30^\circ \)) and, in Fig. 2 (\( \theta = 90^\circ \) and \( 45^\circ \)) and Fig. 3 (\( \theta = 30^\circ \)), for the hot jet. For the data shown, the refraction correction was applied for the cold jet simulation at \( 30^\circ \) and, for the hot one, at \( 45^\circ \) and \( 30^\circ \).

The agreement for the cold jet is quite promising, being verified for all angles in the Mach number range tested. The results show that both the high and the low frequency solutions are relevant for the comparison. For \( \theta = 30^\circ \), the model predicts, in some cases, a lower sound pressure level at the higher frequencies than that verified at \( 90^\circ \), which is possibly due to the use of Eq. (7), obtained from the isotropic source assumption, combined with the refraction modelling.

For hot jets, the agreement is also good, at least up to \( \theta = 45^\circ \). For \( \theta = 30^\circ \), however, the model overpredicts the spectra of the hot jet, mostly for mid and high frequencies, in the full Mach number range considered (only the higher Mach number data, which shows poorer agreement, are shown in Fig. 3). It should be noted that non-compactness, not considered here, should be more effective at low angles.
5. Conclusion

A simple point source model, given by analytical expressions based on asymptotic solutions of Lilley’s equation, is used to predict the spectra of cold and hot jets based on the measured spectrum of the cold jet at $\theta = 90^\circ$. The model contains a set of constants to be adjusted and achieved good to reasonable success, except for hot jets at $\theta = 30^\circ$. Some of the model adjustable constants were found to be different for the hot and cold jets, but, given the amount of approximations and uncertainties involved, this appears as reasonable, implying that the constants are being considered as temperature dependant. The model needs further investigation, as a number of details can be improved upon.

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REFERENCES


![Figure 1](image.png)

Figure 1 (part 1): Unheated jet noise third octave band sound pressure level for different jet mach numbers, $M_J$. Measured data [10]: • $\theta = 45^\circ$ or $30^\circ$; ■ $\theta = 90^\circ$; prediction for $\theta = 45^\circ$ and $30^\circ$: ♦ full solution; --- low frequency solution; ---- high frequency solution.
Figure 1 (part 2): Unheated jet noise third octave band sound pressure level for jet mach number $M_J = 0.9$.
Measured data [10]: ● $\theta = 45^\circ$ or $30^\circ$; ■ $\theta = 90^\circ$; prediction for $\theta = 45^\circ$ and $30^\circ$: ● full solution; -- -- low frequency solution; --- high frequency solution.

Figure 2 (part 1): Jet noise third octave band sound pressure level for different jet mach numbers, $M_J$.
Measured data [10]: ● hot jet ($R = 2.9$) for $\theta = 90^\circ$ or $45^\circ$; ■ cold jet, $\theta = 90^\circ$. Prediction, hot jet, $\theta = 90^\circ$ and $45^\circ$: ● full solution; -- -- low frequency solution; --- high frequency solution.
Figure 2 (part 2): Jet noise third octave band sound pressure level for different jet mach numbers, $M_J$.
Measured data [10]: • hot jet ($R = 2.9$) for $\theta = 90^\circ$ or $45^\circ$; ■ cold jet, $\theta = 90^\circ$. Prediction, hot jet, $\theta = 90^\circ$ and $45^\circ$: • full solution; – – low frequency solution; – – – high frequency solution.

Figure 3: Jet noise third octave band sound pressure level for $M_J = 0.7$ and $0.9$.
Measured data [10]: • hot jet ($R = 2.9$) for $\theta = 30^\circ$; ■ cold jet, $\theta = 90^\circ$. Prediction, hot jet, $\theta = 30^\circ$: • full solution; – – low frequency solution; – – – high frequency solution.