Fiber reinforced composites are used to fabricate components for aerospace, automotive and marine structures because of their high strength and low density. Modeling the vibration transmission in such structures is used as a design tool to ensure vibration and acoustic levels are within prescribed limits. In this work the high frequency dynamics of composite beams are investigated to determine if such structures can be appropriately included as subsystems in the tool, Statistical Energy Analysis. An underlying assumption in Statistical Energy Analysis is that there is an equipartition of modal energy in each subsystem. The study derives expressions for the total modal energy of composite beams with the following layups: (1) symmetric unbalanced layup coupled in bending-torsion and (2) non-symmetric cross-ply layup elastically coupled in bending-extension. It is shown that when beams of both layups are excited by spatial white noise the variation in modal energy over the modal spectrum is non-uniform even under the assumption of a constant damping bandwidth. The behavior is observed to be more pronounced for the laminate coupled in bending-torsion. Expressions for the power flow of the propagating waves on the beam are derived for both lay-ups and it is shown that the strength of the cross-conversion of wavetypes at the boundaries dictate the distribution of modal energy levels.

Keywords: modal energy, statistical energy analysis, fiber reinforced composites, random vibration.

1. Introduction

Fiber reinforced composites (FRCs) are often the preferred materials in industries where low weight and high strength are required. As a result, FRCs may be incorporated into aerospace, automotive and marine structures. The overall layup of a FRC determines the types of couplings present and gives an additional dimension to the their dynamic analysis compared to isotropic structures. Notably, much research has been focused on the elastic couplings present in long slender beam-like structures. Ref. [1] derived the equations of motion of thin walled closed cross section beams, which enabled the free vibration analysis of structures with layups such as the circumferentially asymmetric stiff configuration coupled in bending and torsion [2]. Ref. [3] derived similar equations of motion for flat composite beams coupled in bending and torsion taking the effects of shear and rotary inertia into account. Methods to solve the differential equations
of motion for beams coupled in bending and torsion include the dynamic stiffness approach [4] and a wave approach [5]. Ref. [6] derived the equations of motion for non-symmetric cross ply laminates coupled in bending and extension and calculated non-dimensional natural frequencies for a range of boundary conditions. The forced vibration analysis of FRCs have been analyzed for deterministic and random vibrations using the normal mode approach [7-8].

The majority of the literature has focused on the free and forced vibration characteristics of FRCs; which is useful when the substructure can be accurately modelled with the appropriate boundary conditions as an isolated component. In the high frequency vibroacoustic analysis of complex structures, where a FRC may form a substructural component of the overall system; modeling the component in isolation is not sufficient as it provides limited information. The most common high frequency modeling technique, Statistical Energy Analysis (SEA) analyzes the time and frequency averaged power flow between groups of modes of contiguous substructures. Each group of modes is called a subsystem and a major assumption is that there is an equipartition of energy; that is each mode has the same modal energy within the band [9]. This study analyzes if the criteria of energy equipartition is achieved when an FRC beam coupled in bending-torsion and another coupled in bending-extension are excited by spatial white noise. The results presented are important in determining if FRC beams can be appropriately included in a SEA model.

2. Equations of motion

Two rectangular laminated composite beams with orthogonal axes have origins coincident with the associated beam axis and are as shown in Figs. 1a-b. The beam in Fig. 1a, from hereon called Beam 1, has a symmetric unbalanced layup \([\tilde{\alpha}_o]_{\tilde{\alpha}}\) whilst the beam in Fig. 1b, from hereon called Beam 2, is a non-symmetric cross-ply \([90^o/0^o]_{\tilde{\alpha}}\) laminate. Due to the anisotropic nature of both beams; Beam 1 is coupled in bending and torsion and Beam 2 is coupled in bending and extension. The differential equations of motion for Beam 1 and Beam 2 which include the effects of shear and rotary inertia are derived in Ref. [3] and Ref. [6] respectively. Since both sets of equations are observed to have the same mathematical form; the equations of motion for both beam types, including viscous damping effects and excitation sources can be concisely represented as [8]

\[
\langle EI \rangle_{eq} \frac{\partial^2 \theta}{\partial y^2} + \langle \kappa AG \rangle_{eq} \left( \frac{\partial w}{\partial y} - \theta \right) + \langle \Lambda \rangle_{eq} \frac{\partial^2 \phi}{\partial y^2} - c_1 \dot{\theta} - \rho I \ddot{\theta} = m(y,t), \tag{1a}
\]

\[
\langle \kappa AG \rangle_{eq} \left( \frac{\partial^2 w}{\partial y^2} - \frac{\partial \theta}{\partial y} \right) - c_2 \dot{w} - \rho A \ddot{w} = f(y,t), \tag{1b}
\]

\[
\langle \Omega \rangle_{eq} \frac{\partial^2 \phi}{\partial y^2} + \langle \Lambda \rangle_{eq} \frac{\partial^2 \theta}{\partial y^2} - c_3 \dot{\phi} - \Gamma \ddot{\phi} = g(y,t), \tag{1c}
\]

where \(t\) is time, \(y\) is the coordinate along the length of the beam, \(w\) is the lateral displacement, \(\theta\) is the slope due to bending, \(A = h\epsilon\) is the cross-sectional area, \(h\) is the thickness, \(\epsilon\) is the width, \(I\) is the second moment of area of the cross-section about the \(y\) axis, \(\rho\) is the density of the material, \(\left( \frac{\partial w}{\partial y} - \theta \right)\) is the shear angle, the term \(\phi;\) is either the torsional rotation (\(\psi\)) when considering Beam 1, or the axial displacement (\(v\)) when considering Beam 2 and \(\langle \Lambda \rangle_{eq};\) is either the polar mass moment of inertia (\(I_{\alpha}\)) when considering Beam 1, or the mass per unit length (\(\rho A\)) when considering Beam 2. The overhead (\(\dot{}\)) is the differential with respect to time and the damping coefficients per unit length in bending and rotation are \(c_1\) and \(c_2\) respectively. The damping term \(c_3\); is either the torsional or axial damping coefficient for Beam 1 or Beam 2 respectively. The function \(m(y,t)\) and \(f(y,t)\) are the moment and transverse force per unit length for both beams respectively whilst \(g(y,t)\); is either a torsional load per unit length for Beam 1 or a dynamic axial
load per unit length for Beam 2. For both Beam 1 and Beam 2, \( \langle EI \rangle_{eq} \) and \( \langle \kappa AG \rangle_{eq} \) are the bending and shear rigidities respectively. The term \( \langle \Omega \rangle_{eq} \); is either the twisting rigidity \( \langle GJ \rangle_{eq} \) for Beam 1, or the the extensional rigidity \( \langle EA \rangle_{eq} \) for Beam 2, and \( \langle \Lambda \rangle_{eq} \); is either the bending-twist coupling rigidity \( \langle K \rangle_{eq} \) for Beam 1, or the bending-extension coupling rigidity \( \langle B \rangle_{eq} \) for Beam 2. The rigidity is dependent on the layup of the composite and can be calculated using expressions derived in Ref. [10].

\[ \text{(a) Symmetric laminate with unbalanced layup } [\alpha^0]_n. \]

\[ \text{(b) Non-symmetric cross-ply laminate with layup } [90^o/0^o]_n. \]

Figure 1: Coordinate system for a symmetric unbalanced and non-symmetric cross-ply laminated beam having \( n \) equally thick layers.

### 3. Free vibration

The undamped free vibration characteristics are obtained from Eqs. (1a-c) by letting \( c_1 = c_2 = c_3 = f(y, t) = g(y, t) = m(y, t) = 0 \). The modal response is assumed to be of the form

\[ \theta_n(y, t) = \Theta_n(\xi) e^{i\omega_n t}, \quad w_n(y, t) = W_n(\xi) e^{i\omega_n t}, \quad \phi_n(y, t) = \Phi_n(\xi) e^{i\omega_n t}, \quad (2a-c) \]

where \( L \) is the length of the beam, \( \xi = y/L \) is the non-dimensional length and for the \( n^{th} \) mode shape, \( \omega_n \) is the natural frequency, \( \Theta_n \) is the rotational mode, \( W_n \) is the lateral mode and \( \Phi_n \); is either the torsional mode (\( \Psi_n \)) for Beam 1 or the axial mode (\( V_n \)) for Beam 2. Substituting Eqs. (2a-c) into Eqs. (1a-c) and letting \( y \rightarrow \xi \) the equations of motion are

\[ \frac{\langle EI \rangle_{eq}}{L^2} \frac{d^2 \Theta_n}{d\xi^2} + \frac{\langle \kappa AG \rangle_{eq}}{\kappa} \left( \frac{1}{L} \frac{dW_n}{d\xi} - \Theta_n \right) + \frac{\langle \Lambda \rangle_{eq}}{L^2} \frac{d^2 \Phi_n}{d\xi^2} + \rho I \omega_n^2 \Theta_n = 0, \quad (3a) \]

\[ \frac{\langle \kappa AG \rangle_{eq}}{L^2} \left( \frac{1}{L^2} \frac{d^2 W_n}{d\xi^2} - \frac{1}{L} \frac{d\Theta_n}{d\xi} \right) + \rho A \omega_n^2 W_n = 0 \quad \frac{\langle \Omega \rangle_{eq}}{L^2} \frac{d^2 \Phi_n}{d\xi^2} + \frac{\langle \Lambda \rangle_{eq}}{L^2} \frac{d^2 \Theta_n}{d\xi^2} + \Gamma \omega_n^2 \Phi_n = 0. \quad (3b-c) \]

The solutions for Eqs. (3) have been studied by Refs. [5-6]. When the solution is interpreted from a traveling wave perspective it can be shown that the nature of the waves depends on the whether the natural frequency is less than or greater than \( \omega_{cr} = \sqrt{\langle \kappa AG \rangle_{eq}/\rho L} \). In the case where \( \omega_n < \omega_{cr} \) there are two pairs of propagating waves and one pair of decaying evanescent waves. The mode shapes take the form

\[ \Theta_n(\xi) = \tilde{a}_1 e^{-i\xi} + \tilde{b}_2 e^{-i\xi} + \tilde{a}_3 e^{-s_3\xi} + \tilde{a}_4 e^{i\xi} + \tilde{a}_5 e^{i\xi} + \tilde{a}_6 e^{s_3\xi}, \]

\[ W_n(\xi) = \tilde{b}_1 e^{-i\xi} + \tilde{b}_2 e^{-i\xi} + \tilde{b}_3 e^{-s_3\xi} + \tilde{b}_4 e^{i\xi} + \tilde{b}_5 e^{i\xi} + \tilde{b}_6 e^{s_3\xi}, \]

\[ \Phi_n(\xi) = \tilde{c}_1 e^{-i\xi} + \tilde{c}_2 e^{-i\xi} + \tilde{c}_3 e^{-s_3\xi} + \tilde{c}_4 e^{i\xi} + \tilde{c}_5 e^{i\xi} + \tilde{c}_6 e^{s_3\xi}, \quad (4a-c) \]
where for $i = 1, 2, \ldots, 6$, $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$ are amplitudes and for $j = 1, 2, 3$, $\sqrt{s_j}$ is the non-dimensional wavenumber. The expression for the wavenumbers are [11]

$$s_1 = \sqrt{|n + q \cos(\gamma)|}, \quad s_2 = \sqrt{|n + q \cos\left(\frac{2\pi}{3} - \gamma\right)|}, \quad s_3 = \sqrt{|n + q \cos\left(\frac{2\pi}{3} + \gamma\right)|},$$

(5a-c)

where

$$\tilde{n} = -\frac{b}{3a}, \quad \tilde{q} = \frac{2}{3a}\sqrt{b^2 - 3ac}, \quad \gamma = \frac{1}{3}\cos^{-1}\left(-\tilde{F}\right), \quad \tilde{F} = \frac{4}{a\tilde{q}^3}\left(a\tilde{n}^3 + b\tilde{n}^2 + c\tilde{n} + d\right),$$

$$a = 1 - \frac{\langle \Lambda \rangle^2_{eq}}{\langle EI \rangle_{eq} \langle \Omega \rangle_{eq}}, \quad b = -\frac{L^2\omega_n^2(-\rho A\langle \Lambda \rangle_{eq} + \rho I\langle \kappa AG\rangle_{eq}\langle \Omega \rangle_{eq} + \langle EI \rangle_{eq}\langle \kappa AG\rangle_{eq}\langle \Omega \rangle_{eq})}{\langle EI \rangle_{eq}\langle \kappa AG\rangle_{eq}\langle \Omega \rangle_{eq}},$$

$$c = \frac{L^4\omega_n^2(\Gamma A\omega_n^2\langle EI \rangle_{eq} + \rho A\rho I\omega_n^2\langle \Omega \rangle_{eq} + \langle \kappa AG\rangle_{eq}\langle \Gamma I\omega_n^2 - \rho A\langle \Omega \rangle_{eq}\rangle_{eq}}{\langle EI \rangle_{eq}\langle \kappa AG\rangle_{eq}\langle \Omega \rangle_{eq}},$$

and

$$d = \frac{L^6\Gamma A\omega_n^4(-\rho I\omega_n^2 + \langle \kappa AG\rangle_{eq})}{\langle EI \rangle_{eq}\langle \kappa AG\rangle_{eq}\langle \Omega \rangle_{eq}}.$$ 

(6a-h)

The amplitudes in Eqs. (4a-c) are related by the expressions

$$b^+ = \Pi^+ a^+, \quad b^- = -\Pi^+ a^-, \quad c^+ = \Pi^- a^+ \quad \text{and} \quad c^- = \Pi^- a^-,$$

(7a-d)

where

$$a^+ = \left\{ \tilde{a}_1 \quad \tilde{a}_2 \quad \tilde{a}_3 \right\}^T, \quad a^- = \left\{ \tilde{a}_4 \quad \tilde{a}_5 \quad \tilde{a}_6 \right\}^T, \quad b^+ = \left\{ \tilde{b}_1 \quad \tilde{b}_2 \quad \tilde{b}_3 \right\}^T,$$

$$b^- = \left\{ \tilde{b}_4 \quad \tilde{b}_5 \quad \tilde{b}_6 \right\}^T, \quad c^+ = \left\{ \tilde{c}_1 \quad \tilde{c}_2 \quad \tilde{c}_3 \right\}^T, \quad c^- = \left\{ \tilde{c}_4 \quad \tilde{c}_5 \quad \tilde{c}_6 \right\}^T,$$

$$\Pi^+ = \text{diag}\left[ \frac{s_1^2(\lambda)_{eq}}{-L^2\rho A\omega_n^4 + s_1^2(\kappa AG)_{eq}}, \quad \frac{s_2^2(\lambda)_{eq}}{-L^2\rho A\omega_n^4 + s_2^2(\kappa AG)_{eq}}, \quad \frac{Ls_3(\lambda)_{eq}}{L^2\rho A\omega_n^4 + s_2^2(\kappa AG)_{eq}} \right],$$

and

$$\Pi^- = \text{diag}\left[ \frac{s_2^2(\lambda)_{eq}}{L^2\omega_n^2 - s_2^2(\lambda)_{eq}}, \quad \frac{s_1^2(\lambda)_{eq}}{L^2\omega_n^2 - s_1^2(\lambda)_{eq}}, \quad -\frac{s_3^2(\lambda)_{eq}}{L^2\omega_n^2 - s_2^2(\lambda)_{eq}} \right].$$

(8a-h)

Eqs. (8g-h) are a pair of diagonal $3 \times 3$ matrices. The frequencies above $\omega_c$ are not considered in this work as they are very large for practical beams and of little concern to noise and vibration engineers.

The orthogonality condition for the natural modes of vibration are [8]

$$\int_0^1 (\rho I\Theta_m\Theta_n + \rho A W_m W_n + \Gamma \Phi_m \Phi_n)\,d\xi = \mu_n \delta_{mn},$$

(9)

where $\mu_n$ is the generalized mass and $\delta_{mn}$ is the Kronecker delta function. The natural frequencies and hence the mode shapes may be readily evaluated using the principle of wave train phase closure for known boundary conditions [5].
4. Power analysis

The nature of the waves when \( \omega_n < \omega_{cs} \) implies that when a single propagating wave impinges on a boundary two propagating wave-types \((s_1, s_2)\) are reflected. The time averaged power \( P_i \) of a propagating wave-type \((i)\) is,

\[
P_i = \frac{\omega_n}{2\pi} \int_0^{2\pi/\omega} \left( \text{Re} \left\{ M_y \right\} \text{Re} \left\{ \phi \right\} + \text{Re} \left\{ N_y \right\} \text{Re} \left\{ \phi \right\} + \text{Re} \left\{ Q_{yz} \right\} \text{Re} \left\{ \phi \right\} \right) dt \tag{10}
\]

where

\[
M_y = \frac{\langle EI \rangle_{eq} \partial \theta}{L} + \frac{\langle A \rangle_{eq} \partial \phi}{L}, \quad N_y = \frac{\langle \Omega \rangle_{eq} \partial \theta}{L}, \quad Q_{yz} = \langle \kappa AG \rangle_{eq} \left( \frac{1}{L} \partial w - \theta \right). \tag{11a-c}
\]

The terms \( M_y \) and \( Q_{yz} \) are the bending moment and shear force respectively whilst \( N_y \) is the torque \((T_y)\) when dealing with Beam 1 or the axial force \((P_y)\) when dealing with Beam 2 \([5-6]\). Therefore considering an arbitrary propagating wave of the form

\[
\theta_i(\xi, t) = \tilde{\alpha}_i e^{i(\omega_n t - s_i \xi)}, \quad w_i(\xi, t) = \tilde{b}_i e^{i(\omega_n t - s_i \xi)} , \quad \phi_i(\xi, t) = \tilde{c}_i e^{i(\omega_n t - s_i \xi)}, \tag{12a-c}
\]

where \( i = 1, 2 \) and the complex amplitudes are \( \tilde{\alpha}_i, \tilde{b}_i, \tilde{c}_i \) are related by Eqs. \( 7a \) and \( 7c \); it follows upon substitution of Eqs. \((12a-c)\) into Eq. \((10)\)

\[
P_i = \frac{\omega_n s_i}{2L} \left( \langle EI \rangle_{eq} + \frac{\rho A L^4 \omega_n^2 \langle \kappa AG \rangle_{eq}^2}{\left( \rho A L^2 \omega_n^2 - s_i^2 \langle \kappa AG \rangle_{eq} \right)^2} + \frac{s_i^2 \langle \lambda \rangle_{eq}^2 \left( 2L^2 \Gamma \omega_n^3 - s_i^2 \langle \Omega \rangle_{eq} \right)}{\left( L^2 \Gamma \omega_n^3 - s_i^2 \langle \Omega \rangle_{eq} \right)^2} \right) |\tilde{\alpha}_i|^2. \tag{13}
\]

Turning our attention to the power transmitted across a boundary we can consider the reflection matrix for any arbitrary boundary condition for Beam 1 or 2 neglecting the evanescent waves as \([11]\)

\[
\begin{bmatrix}
\tilde{\alpha}_4 \\
\tilde{\alpha}_5 
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
\tilde{\alpha}_1 \\
\tilde{\alpha}_2
\end{bmatrix}, \tag{14}
\]

Eq. \((13)\) can now be used to calculate the power of a reflected wave \( (j) \) due to an impinging wave \( (i) \) at the boundary interface using,

\[
P_{ij} = \frac{\omega_n s_j}{2L} \left( \langle EI \rangle_{eq} + \frac{\rho A L^4 \omega_n^2 \langle \kappa AG \rangle_{eq}^2}{\left( \rho A L^2 \omega_n^2 - s_j^2 \langle \kappa AG \rangle_{eq} \right)^2} + \frac{s_j^2 \langle \lambda \rangle_{eq}^2 \left( 2L^2 \Gamma \omega_n^3 - s_j^2 \langle \Omega \rangle_{eq} \right)}{\left( L^2 \Gamma \omega_n^3 - s_j^2 \langle \Omega \rangle_{eq} \right)^2} \right) |a_{ji}|^2, \tag{15}
\]

where \( i, j = 1, 2 \). Combining Eqs. \((15)\) and \((13)\) it can be shown for arbitrary boundary conditions the ratio of the power of a reflected outgoing wave to the incident wave across a boundary may be concisely represented as

\[
\frac{P_{11}}{P_1} = |a_{11}|^2, \quad \frac{P_{12}}{P_1} = 1 - |a_{11}|^2, \quad \frac{P_{22}}{P_2} = |a_{22}|^2, \quad \frac{P_{21}}{P_2} = 1 - |a_{22}|^2. \tag{16a-d}
\]

Eqs. \((16)\) demonstrate that only the diagonal terms of the reflection matrix of Eq. \((14)\) are required to calculate the fractional powers.
5. Random vibration

The free vibration analysis discussed in the previous section identified the natural modes of vibration which will now be used to calculate the forced response and modal energy levels of the beam. Assuming the forced vibration response is a summation of the contributions of the individual modes then

\[
\theta (\xi, t) = \sum_{n=1}^{\infty} \Theta_n (\xi) q_n (t), \quad w (\xi, t) = \sum_{n=1}^{\infty} W_n (\xi) q_n (t), \quad \phi (\xi, t) = \sum_{n=1}^{\infty} \Phi_n (\xi) q_n (t). \quad (17a-c)
\]

Substituting the assumed solutions Eqs. (17a-c) into Eqs. (1a-c) and using the results obtained in Eqs. (3a-c) results in

\[
\sum_{n=1}^{\infty} \left( \rho I \Theta_n \ddot{q}_n + c_1 \Theta_n \dot{q}_n + \omega_n^2 \rho I \Theta_n q_n \right) = m (\xi, t), \quad \sum_{n=1}^{\infty} \left( \rho AW_n \ddot{q}_n + c_2 W_n \dot{q}_n + \omega_n^2 \rho AW_n q_n \right) = f (\xi, t), \quad (18a-c)
\]

Next we multiply Eqs. (18a-c) by \( \Theta_m \), \( W_m \) and \( \Phi_m \) respectively and sum the resulting equations. Upon the integrating the summation result over \( \xi \) and applying Eq. (9) the resulting equation is

\[
\ddot{q}_n + \omega_n \eta_n \dot{q}_n + \omega_n^2 \Gamma q_n = M_n (t) + F_n (t) + G_n (t), \quad (19)
\]

where

\[
M_n (t) = \frac{1}{\mu_n} \int_{0}^{1} \Theta_n (\xi) m (\xi, t) \, d\xi, \quad F_n (t) = \frac{1}{\mu_n} \int_{0}^{1} W_n (\xi) f (\xi, t) \, d\xi,
\]

\[
G_n (t) = \frac{1}{\mu_n} \int_{0}^{1} \Phi_n (\xi) g (\xi, t) \, d\xi \quad \text{and} \quad \eta_n = \frac{c_1}{\rho I \omega_n} = \frac{c_2}{\rho A \omega_n} = \frac{c_3}{\Gamma \omega_n}. \quad (20a-d)
\]

The terms \( M_n \), \( F_n \) and \( G_n \) are the modal excitation forces and \( \eta_n \) is the damping loss factor. The relationship assumed for \( \eta_n \) in Eq. (20d) ensures \( \dot{q}_n \) in Eq. (19) is decoupled after applying the orthogonality condition (Eq. (9)).

For a unit harmonic point excitation applied at \( \xi = \xi_o \) the forcing functions are \( m (\xi, t) = f (\xi, t) = g (\xi, t) = e^{\imath \omega t} \delta (\xi - \xi_o) \); which results in

\[
M_n (t) = \frac{1}{\mu_n} \Theta_n (\xi_o) e^{\imath \omega t}, \quad F_n (t) = \frac{1}{\mu_n} W_n (\xi_o) e^{\imath \omega t}, \quad G_n (t) = \frac{1}{\mu_n} \Phi_n (\xi_o) e^{\imath \omega t}. \quad (21a-c)
\]

Upon assuming that the solution for Eq. (19) is of the form

\[
q_n (t) = H_n (\omega) (\Theta_n (\xi_o) + W_n (\xi_o) + \Phi_n (\xi_o)) e^{\imath \omega t}, \quad (22)
\]

then substituting Eqs. (21-22) into Eq. (19) leads to

\[
H_n (\omega) = \frac{1}{\mu_n (\omega_n^2 - \omega^2 + \imath \eta_n \omega)}, \quad (23)
\]

which is the complex frequency response function.

We now consider the entire length of the beam to be excited by spatial white noise or rain-on-the-roof excitation such that \( S_{OM}, S_{OF} \) and \( S_{OG} \) are the power spectral densities of the moment,
lateral and torsional (or axial) excitations respectively. Rain-on-the-roof excitation has a delta correlated cross spectral density,\
\[ R_f (\xi, \tau) = \langle \tilde{f} (\xi, t) \rangle \langle \tilde{f} (\xi + \xi_o, t + \tau) \rangle = \delta (\xi_o) \delta (\tau) S_o, \]  
(24)

where \( \langle \rangle \) is the ensemble average of the random process, \( \tilde{f} \) is the excitation being considered, \( R_f \) is the cross-correlation function, and \( S_o \) is the constant spectral density which may be \( S_{OM}, S_{OF} \) or \( S_{OG} \) in our case. This type of forcing field allows all modes to be subjected to the same level of excitation and no particular mode or group of modes are favoured.

It can be shown that by modifying the results obtained by Ref. [12] the spectral densities for the response of Beam 1 or Beam 2 due to white noise are given by,

\[
\left\{ \begin{array}{l}
S_{\Theta} \\
S_{W} \\
S_{\Phi}
\end{array} \right\} = \begin{bmatrix}
\int H_{11}^* H_{13} \, d\xi_o & \int H_{12}^* H_{14} \, d\xi_o & \int H_{13}^* H_{15} \, d\xi_o \\
\int H_{21}^* H_{23} \, d\xi_o & \int H_{22}^* H_{24} \, d\xi_o & \int H_{23}^* H_{25} \, d\xi_o \\
\int H_{31}^* H_{33} \, d\xi_o & \int H_{32}^* H_{34} \, d\xi_o & \int H_{33}^* H_{35} \, d\xi_o
\end{bmatrix}
\left\{ \begin{array}{l}
S_{OM} \\
S_{OF} \\
S_{OG}
\end{array} \right\},
\]  
(25)

where the superscript (*) denotes the complex conjugate frequency response function and

\[
H_{11} = \sum_{n=1}^{\infty} \Theta_n (\xi_o) H_n (\omega) \Theta_n (\xi), \quad H_{12} = \sum_{n=1}^{\infty} W_n (\xi_o) H_n (\omega) \Theta_n (\xi), \quad H_{13} = \sum_{n=1}^{\infty} \Phi_n (\xi_o) H_n (\omega) \Theta_n (\xi),
\]

\[
H_{21} = \sum_{n=1}^{\infty} \Theta_n (\xi_o) H_n (\omega) W_n (\xi), \quad H_{22} = \sum_{n=1}^{\infty} W_n (\xi_o) H_n (\omega) W_n (\xi), \quad H_{23} = \sum_{n=1}^{\infty} \Phi_n (\xi_o) H_n (\omega) W_n (\xi),
\]

\[
H_{31} = \sum_{n=1}^{\infty} \Theta_n (\xi_o) H_n (\omega) \Phi_n (\xi), \quad H_{32} = \sum_{n=1}^{\infty} W_n (\xi_o) H_n (\omega) \Phi_n (\xi), \quad H_{33} = \sum_{n=1}^{\infty} \Phi_n (\xi_o) H_n (\omega) \Phi_n (\xi).
\]  
(26a-i)

In Eqs. (25) \( S_W \) and \( S_{\Theta} \) are the spectral densities of the lateral and rotational displacements respectively and \( S_{\Phi} \) is the spectral density of the torsional displacement for Beam 1 or the axial displacement for Beam 2. The mean square velocity levels may then be calculated using [12]

\[
\langle \theta^2 \rangle = \int_{-\infty}^{\infty} \omega^2 S_{\Theta} \, d\omega, \quad \langle \dot{\theta}^2 \rangle = \int_{-\infty}^{\infty} \omega^2 S_W \, d\omega, \quad \langle \dot{\phi}^2 \rangle = \int_{-\infty}^{\infty} \omega^2 S_{\Phi} \, d\omega.
\]  
(27a-c)

The spectral densities of the response in Eq. (25) are evaluated by recognizing that the product \( H_n (\omega) H_n^* (\omega) = |H_n (\omega)|^2 \) and using the following results from Ref. [13]

\[
\int_0^{\infty} |H_n (\omega)|^2 \, d\omega = \frac{\pi}{2 \mu_n^2 (\eta_n \omega_n)}, \quad \int_0^{\infty} |\omega H_n (\omega)|^2 \, d\omega = \frac{\pi}{2 \mu_n^2 (\eta_n \omega_n)}. \]  
(28a-b)

It follows the mean square responses may now be expressed as

\[
\langle \theta^2 \rangle = \sum_{n=1}^{\infty} \frac{\pi \Theta_n (\xi_o) \Theta_n (\xi_o) P}{\mu_n^2 (\eta_n \omega_n)}, \quad \langle \dot{\theta}^2 \rangle = \sum_{n=1}^{\infty} \frac{\pi W_n (\xi_o) W_n (\xi_o) \tilde{P}}{\mu_n^2 (\eta_n \omega_n)}, \quad \langle \dot{\phi}^2 \rangle = \sum_{n=1}^{\infty} \frac{\pi \Phi_n (\xi_o) \Phi_n (\xi_o) \tilde{P}}{\mu_n^2 (\eta_n \omega_n)}, \]  
(29a-c)

where

\[
\tilde{P} = S_{OM} \int_0^{1} \Theta_n (\xi_o) \Theta_n (\xi_o) \, d\xi_o + S_{OF} \int_0^{1} W_n (\xi_o) W_n (\xi_o) \, d\xi_o + S_{OG} \int_0^{1} \Phi_n (\xi_o) \Phi_n (\xi_o) \, d\xi_o.
\]  
(30)

The term \( \tilde{P} \) consists of contributions from all the excitation sources and is termed the total modal excitation.
6. Modal energy

The expected total energy of the beam for the $n^{th}$ mode ($\langle E_n \rangle$) in terms of the velocities are

$$\langle E_n \rangle = \int_0^1 \left( \rho I \langle \dot{\theta}^2 \rangle_n + \rho A \langle \dot{w}^2 \rangle_n + \Gamma \langle \dot{\phi}^2 \rangle_n \right) \, d\xi.$$  \hspace{1cm} (31)

Substituting Eqs. (29a-c) in Eq. (31) and using the orthogonality relationship in Eq. (9) the expression simplifies to

$$\langle E_n \rangle = \frac{\pi \bar{P}}{\mu_n (\eta_n \omega_n)}.$$ \hspace{1cm} (32)

When the half power bandwidth ($\Delta = \eta_n \omega_n$) is considered constant the modal energy is now dependent on $\bar{P}/\mu_n$, which suggests that generally there is an uneven distribution of modal energy across successive modes. This is in sharp contrast to isotropic beams which have uniform modal energy levels when subjected to rain on the roof excitation under the assumption of a constant bandwidth [12]. For the specific cases of the rectangular sectioned symmetric and cross ply laminates; the modal energy for either Beam 1 or Beam 2 from Eq. (32) can be expressed as

$$\langle E_{n,Beam} \rangle = \frac{\pi \bar{P}_{n,Beam}}{\rho A \Delta \bar{\mu}_{n,Beam}}, \quad i = 1, 2$$ \hspace{1cm} (33)

where

\begin{align*}
\bar{P}_{n,Beam1} &= S_{OM} \int_0^1 \Theta_n(\xi_o) \Theta_n(\xi_o) d\xi_o + S_{OF} \int_0^1 W_n(\xi_o) W_n(\xi_o) d\xi_o + S_{OT} \int_0^1 \Psi_n(\xi_o) \Psi_n(\xi_o) d\xi_o, \\
\bar{\mu}_{n,Beam1} &= \int_0^1 \left( \frac{h^2}{12} \Theta_n(\xi) \Theta_n(\xi) + W_n(\xi) W_n(\xi) + \left( \frac{h^2 + \epsilon^2}{12} \right) \Psi_n(\xi) \Psi_n(\xi) \right) d\xi, \\
\bar{P}_{n,Beam2} &= S_{OM} \int_0^1 \Theta_n(\xi_o) \Theta_n(\xi_o) d\xi_o + S_{OF} \int_0^1 W_n(\xi_o) W_n(\xi_o) d\xi_o + S_{OV} \int_0^1 V_n(\xi_o) V_n(\xi_o) d\xi_o, \\
\bar{\mu}_{n,Beam2} &= \int_0^1 \left( \frac{h^2}{12} \Theta_n(\xi) \Theta_n(\xi) + W_n(\xi) W_n(\xi) + V_n(\xi) V_n(\xi) \right) d\xi. \hspace{1cm} (34a-d)
\end{align*}

In this work the exact expected modal energy level as calculated from Eq. (28) is of less importance compared to the variation in energy across the natural modes. As a result the ratio

$$\langle E_{n,Beam} \rangle \sim \frac{\bar{P}_{n,Beam}}{\bar{\mu}_{n,Beam}},$$ \hspace{1cm} (35)

will be used for the numerical analysis in the subsequent section.
7. Numerical simulations

The numerical calculations are done for two flat beams, Beam 1A and Beam 2A which have similar dimensions, \( h = 0.01m, \epsilon = 5h \) and \( L = 100h \). Both beams are made of AS/3501-6 graphite-epoxy which has base properties \([6,14]\): \( E_1 = 9.6 \times 10^9 \text{N/m}^2, E_2 = 145 \times 10^9 \text{N/m}^2, G_{12} = G_{23} = 4.1 \times 10^9 \text{N/m}^2, G_{13} = 3.4 \times 10^9 \text{N/m}^2, \rho = 1570 \text{kg/m}^3 \) and \( \nu_{21} = 0.3 \). Neither Refs. \([6,14]\) stated whether material properties were obtained from static or dynamic tests. Ref. \([15]\), however, studied a related composite, AS4/3501-6 graphite-epoxy, and determined the material properties from static tests. The base material properties obtained from Ref. \([15]\) aligns well with that used by Refs. \([6,14]\). Ref. \([15]\) further showed that the material properties obtained from static tests were adequate to characterize the bending and dynamic response of the laminate.

The shear correction factor is \( \kappa = 5/6 \) for all calculations. Beam 1A is an unbalanced angle ply AS/3501-6 graphite-epoxy beam with fibre orientation \( \tilde{\alpha} = \pi/12 \text{ radians} \) or \( 15^\circ \). The variations of the rigidities with fibre orientation for the unidirectional composite are calculated using Ref. \([10]\) and are as shown in Figs. 2a-b. Beam 2A is a non-symmetric cross-ply laminate with lay-up \([90^\circ/0^\circ]\) and the rigidities are also calculated using Ref. \([10]\).

![Figure 2: Variation of rigidities with fibre orientation for a balanced angle ply laminate as derived from Ref. \([10]\).](image)

Consider the case where both Beam 1 and Beam 2 have simply supported end conditions and are only subjected to transverse rain-on-the-roof excitation with the constant spectral density taken as unity; therefore \( S_{OM} = S_{OT} = 0 \) and \( S_{OF} = 1 \). The natural frequencies and mode shapes for both beams may be calculated using Refs. \([5-6]\). The normalization \( \int_0^1 W_n(\xi) W_n(\xi) d\xi = 1 \) is employed in Eq. (35) to generate Figs. 3a and b which show the variation in modal energy across successive modes for Beam 1A and 2A respectively. Letting the spectral density \( S_{OF} = 1 \) in Eq. (35) results in the modal energy always being less than unity. This conveniently allows a simple comparison with the power ratios between incident and reflected wave-types \( P_{ii}/P_i \) and \( P_{ij}/P_i \) which must also be less than unity.

From Fig. 3a it is observed that the modal energy distribution of Beam 1A is uneven and has a complex relationship with frequency. Fig. 3b shows that Beam 2A also has a complex relationship, however, the variation of modal energy across the modal spectrum is less pronounced. The modes in Figs. 3a-b can be repartitioned into equal energy ranges that span from low to high. The repartitioned modes for Beam 1A and Beam 2A are shown in Figs. 4a-b respectively. In the case of Beam 1A the majority of the modes are localized in the lowest and highest energy levels; whereas in Beam 2A, the largest concentration of modes are in the highest energy range with the adjacent
Figure 3: Bar chart showing variation of modal energy levels for Beam 1A and Beam 2A when simply supported and subjected to transverse rain-on-the roof loading. The wave transformation across a simply supported end is superimposed on the chart. Legend: bars of modal energy  | , fractional power of reflected $s_i$ wave due to an incident $s_i$ wave  | | , fractional power of reflected $s_i$ wave due to an incident $s_j$ wave  | | $i, j = 1, 2, i \neq j$.

Figure 4: Repartition of the modal energy for simply supported beams subjected rain on the roof excitation.

The energy distribution of both types of composites can be further investigated by considering the means in which power due to the propagating waves is transmitted across the simply supported boundary. Using Eqs. (16a-d) and the diagonal terms of the reflection matrix (as derived from Ref. [5]), the ratio of the reflected to incident wave types across the simply supported junction is calculated and the data superimposed on the modal energy bar charts of Figs. 3a-b. It is observed that for both beams when a $s_i$ wave is converted into a similar $s_i$ wave across the boundary the power relationship is the same regardless if $i = 1$ or $i = 2$. Likewise when a $s_i$ wave is converted into a dissimilar $s_j$ wave ($i \neq j$) across the boundary, the power relationship is also the same whether $i = 1$, $j = 2$ or $i = 2$, $j = 1$.

More consequentially for Beam 1A; the power relations indicate that an incident wave type is almost completely reflected into a similar wave type for the very lowest order modes and as the mode number increases the cross-conversion of a $s_i$ wave-type into a dissimilar $s_j$ wave-type increases up to a point where a $s_i$ wave is then completely converted into a $s_j$ wave. Beyond this point the cross-conversion of a $s_i$ wave into a dissimilar $s_j$ wave gradually decreases and at the highest order modes an incident wave-type is once again almost completely reflected into a similar wave-type. For Beam 2A the power relations indicate that for the entire modal spectrum the cross conversion of a $s_i$ wave-type into a dissimilar $s_j$ wave-type is generally weak.

The power relations correlate directly to the distribution of modal energy. Due to the gener-
ally weak cross conversion of wave-types for the cross ply laminate Fig. 3b appears to be almost equipartitioned for a large group of non-successive modes across the modal spectrum. This behaviour is only seen for the angly ply laminate for the higher order modes. Taking the analysis of Beam 1A one step further; if we view the modes for the beam in Fig. 3a as occupying two separate regions; (1) higher order modes where the cross-conversion of wave-types is weak such that the power ratio \( P_{ii}/P_i \) is greater than 0.95 for all modes and (2) all remaining lower order modes where there may be negligible to strong cross-conversion of wave-types at the boundary. Figs. 5a-b are the re-partitioned modal energy for both regions in Beam 1A. In both regions the modal energy level distribution is still non-uniform, however, based on the results in Figs. 3-5 the power associated with the wave transformation across the simply supported boundary dictates the distribution of the modal energy levels for both beams such that; (1) for Beam 1A when the cross-conversion of wave-types is moderate to strong there are no dominant groups of modes occupying any range of energy levels, whereas when the cross-conversion of wave-types is weak there exists two dominant groups of modes occupying the highest and lowest energy levels, and (2) for Beam 2A the cross-conversion of wave-types is generally weak and a large number of modes with the highest energy levels are near to achieving equipartition.

![Figure 5: Repartition of the modal energy for the simply supported Beam 1A when subjected to rain on the roof excitation.](image)

### 8. Conclusions

In this work the variation of modal energy across the modal spectrum is investigated for fibre reinforced composite beams with a symmetric unbalanced angle ply layup and a non-symmetric cross ply layup. Although the angle ply layup is elastically coupled in bending-torsion and the cross ply layup is coupled in bending-extension, the equations of motion of both beams take the same form and their resulting mathematical analysis are conveniently carried out in parallel. For both beam-types there exists two pairs of dissimilar propagating waves and a pair of decaying waves. The relationship between reflected and incident propagating waves can be represented in a \( 2 \times 2 \) matrix. Expressions for the power flow of the propagating waves across the end boundary of the beam is derived in terms of the diagonal terms of the reflection matrix. Analysis of the forced vibration of both beam-types are undertaken and an expression for the total modal energy is derived. Numerical simulations are presented for both beam-types under the action of rain-on-the-roof excitation. Using the constant damping bandwidth model; it is demonstrated that unlike isotropic beams, the modal energies are not equally partitioned for composite beams with elastic couplings. A power-flow analysis shows that the distribution of the modal energy is dictated by the cross-conversion of wave-types at the boundary. The lack of equipartition in modal energy
means that the traditional way in which SEA subsystems are selected are not appropriate for the composite beam coupled in bending-torsion or bending-extension. The work forms the basis of developing a means by which elastically coupled FRC beams can be incorporated in high frequency vibroacoustic models.

REFERENCES


