INVESTIGATION ON DYNAMIC BEHAVIOR OF A PERIODIC BEAM STRUCTURE CARRYING SPRING-MASS SYSTEM BY DYNAMIC STIFFNESS METHOD

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This paper deals with the investigation of the dynamics of a periodic beam structure carrying multiple spring-mass systems by using dynamic stiffness method (DSM), within which flexural, compressional, and torsional vibrations are taken into account. The local dynamic stiffness matrices for individual beams and spring-mass systems are then assembled into global dynamic stiffness matrices so as to calculate vibration transmission from excitation source to flexible beamlike foundations. The finite element method (FEM) investigation is carried out to validate and demonstrate the accuracy and efficiency of our present formulation in modelling the dynamic behaviour of complex periodic beam structures. In addition, parametric influences on vibration transmission are addressed, including different lumped masses, connecting angles, and lower layer spring arrangements.

Keywords: dynamic stiffness method; periodic beam structure; spring-mass system

1. Introduction

Beam-like structures can be often found in many engineering fields, such as bridges and ship mounting rafts. Beam structures have much more configuration flexibility and better maintainability which can be easily repaired or replaced. Therefore, recently, there is a strong demand to accurately predict the dynamics of beam structures carrying machines.

Up to now, numerous studies have been conducted on the dynamics of beam structures. A variety of methods including differential transform method[1], Rayleigh-Ritz method[2,3], transfer matrix method[4,5], and dynamic stiffness method[6-9], are extensively scattered in the literature.

As a powerful alternative to the traditional analytical methods, the dynamic stiffness method has evoked more and more attentions since it was proposed in the early 1940s. More and more studies on laminated beams with the dynamic stiffness method have scattered, but studies on built-up beam structures or beams with spring-mass systems are relatively fewer. Yildirim[10]used the stiffness method for the solution of in-plane free vibration problem of symmetric cross-ply laminated beams with the effects of rotary inertia, axial and transverse shear deformations included by FSDT. Banerjee[11]derived and solved the governing differential equation of motion of a moving Bernoulli–Euler beam in closed analytical form to develop the dynamic stiffness matrix and then used the matrix to investigate free flexural vibration characteristics of the beam. Li Jun and Hua Hongxing[9]investigated free vibration characteristics of shear deformable elastic beams subjected to different sets of boundary conditions based on a unified one-dimensional shear deformation beam theory by using dynamic stiffness method. Rossit and Laura[11]studied the free vibration problem of a cantilever beam with a spring-mass system attached at the tip. Zhi-Jing Wu and Feng-Ming Li[12]extended the spectral element method (SEM) to analyse the vibration band-gap properties of
The main objective of this paper is to investigate the basic dynamic properties of a periodic built-up beam structure carrying spring-mass systems based on dynamic stiffness formulation. In Section 2, our proposed method is applied to the structure and demonstrated from finite element method. Those solutions are in excellent agreement, which illustrates the great potentials of our method in vibration analysis of complex built-up structures. Besides, different parameters influencing the acceleration level of the whole isolation system are investigated respectively.

2. Brief derivation of dynamic stiffness matrix

As illustrated in Fig.1, there is a built-up beam structure with spring-mass system, in which two beams rigidly joined at their junctions with lumped masses and spring-mass block. To develop the dynamic stiffness matrix, an individual beam or a spring-mass joint element is taken from the whole structure, and the local coordinates oxyz are attached to those elements.

Fig 1 A simplified built-up beam structure carrying spring-mass system.

2.1 Matrix derivation of beam element

Based on the Bernoulli-Euler theory, the derivation of dynamic stiffness matrix of a beam element, relating the amplitudes of forces and moments with displacements and rotations at its nodes, is summarized below. The governing equations of the flexural motion in z-direction of a beam with length \( l \) is given by

\[
EI_y \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0
\]

where \( w(x,t) \) is the transverse displacement in z-direction. \( E \) and \( \rho \) are the Young’s modulus and density. \( EI_y \) is the flexural rigidity in y-direction and \( \rho A \) is the mass per unit length.

Then, the solution can be expressed in the following forms

\[
w(x,t) = \delta_i(x) q(t) \]

The relation between the nodal forces and displacements can be given by

\[
K_u q_u = P_u
\]

where \( q_u = \begin{bmatrix} w_1 & \theta_{z1} & w_2 & \theta_{z2} \end{bmatrix}^T \) is the nodal displacement vector, \( P_u = \begin{bmatrix} f_{z1} & m_{z1} & f_{z1} & m_{z1} \\ f_{z2} & m_{z2} & f_{z2} & m_{z2} \end{bmatrix}^T \) is the nodal force vector, and \( K_u \) is the dynamic stiffness matrix for the bending vibration in the x-z plane which is given in Appendix A. The derivations of the bending component in the x-y plane, the compressional wave, and the torsional wave follow a similar procedure as that of the bending component in the x-z plane. The dynamic matrices of these wave types can be assembled to a complete matrix, that is

\[
P = K \cdot q
\]

with

\[
q = \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{s1} & \theta_{s1} & u_2 & v_2 & w_2 & \theta_{s2} & \theta_{s2} \end{bmatrix}^T
\]

\[
P = \begin{bmatrix} f_{s1} & f_{s1} & f_{s1} & m_{s1} & m_{s1} & m_{s1} & f_{s2} & f_{s2} & m_{s2} & m_{s2} \end{bmatrix}^T
\]
where K is the overall dynamic stiffness matrix of the whole beam freedoms also given in Appendix A.

2.2 Matrix derivation of joint element

Shown in Fig. 2(a) is a mass joint element in local coordinate system. The node with lumped mass m is defined at one of the ends of the beam element. The nodal displacements are given by

\[
q_m = \begin{bmatrix}
u_m & w_m & \theta_{xm} & \theta_{ym}
\end{bmatrix}^T, \quad \text{and the forces and moments are defined as:}
\]

\[
\begin{cases}
f_{zm} = \omega^2 m w_m \\
m_{ym} = \omega^2 J \theta_{ym}
\end{cases}
\]

Thus the dynamic stiffness equation of lumped mass can be obtained:

\[
\begin{bmatrix}
f_{zm} \\
f_{ym} \\
f_{zm} \\
m_{ym} \\
m_{ym}
\end{bmatrix} =
\begin{bmatrix} -\omega^2 m & -\omega^2 m \\
-\omega^2 m & \omega^2 J \\
-\omega^2 m & \omega^2 J \\
m_{ym} & \omega^2 J
\end{bmatrix}
\begin{bmatrix}
u_m \\
w_m \\
\theta_{xm} \\
\theta_{ym}
\end{bmatrix}
\]

Thus the dynamic stiffness equation of lumped mass can be obtained:

\[
\left[
\begin{array}{c}
f_{z1} \\
f_{z2}
\end{array}
\right] =
\left[
\begin{array}{cc}
k_z & -k_z \\
-k_z & k_z - \omega^2 m
\end{array}
\right]
\left[
\begin{array}{c}
w_1 \\
w_2
\end{array}
\right]
\]

2.3 Coordinate transformation of global dynamic stiffness matrix

The dynamic stiffness matrix derived in the previous subsection is expressed in local coordinates. In order to obtain the dynamics of the entire beam structure, local dynamic stiffness matrix for each element shall be firstly transformed into global coordinates. Then the transformed matrices are assembled into overall stiffness matrix in accordance to the layout or the construction of the beam structure.

For element number i, the transformation of the displacements and forces at any nodes from global coordinates to local is obtained by a matrix \(T_i\), which can be expressed as:

\[
T_i = \begin{bmatrix}
\overline{\lambda}_i \\
\overline{\lambda}_i
\end{bmatrix}
\]
Thus for a single element, the relationship between the displacements or forces in global coordinates $\vec{q}_i / \vec{P}_i$ and in local coordinates $q_i / P_i$ is written as,

$$q_i = T_i \cdot \vec{q}_i$$

$$P_i = T_i \cdot \vec{P}_i$$

Therefore, the transformation of stiffness matrix can be expressed as follows,

$$K_i = T^T_i \cdot K_i \cdot T_i$$

where $K_i$ and $\overline{K}_i$ are the dynamic stiffness matrix in local and global coordinates, respectively. Next, the transformed stiffness matrix $\overline{K}_i$ shall be assembled into overall dynamic stiffness matrix by using standard finite element assembly techniques\[13\]. The assembly procedure is the same as that in FEM in which the degrees of freedom for elements correspond to nodes.

### 3. Numerical validation and parameters discussion

In this section, the vibration characteristics of the periodic beam structure carrying spring-mass system as illustrated in Fig.3 are studied. The geometry and material parameters are shown in Table 1. The disturbance $F = F_0 e^{i\omega t}$ is located with $F_0 = 1N$. There are four response points A, B, C, D, which located at the machine’s mounting, on the beam raft, under the beam raft and at the foundation beam, respectively. By solving dynamic equations, the frequency responses at those points can be obtained.

#### Table 1 Geometry and material parameters

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$r$</th>
<th>$E$</th>
<th>$\rho$</th>
<th>$\eta$</th>
<th>$m_{\text{lumped}}$</th>
<th>$m_{\text{machine}}$</th>
<th>$k_{\text{upper}}$</th>
<th>$k_{\text{lower}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3m</td>
<td>0.45m</td>
<td>0.01m</td>
<td>210GPa</td>
<td>7850kg/m$^3$</td>
<td>0.01</td>
<td>0.2kg</td>
<td>50kg</td>
<td>200kN/m</td>
<td>350kN/m</td>
</tr>
</tbody>
</table>

#### 3.1 Validation: Dynamic responses due to unit concentrated harmonic forces

To validate our method in forced vibration analysis, the FEM model is developed with mesh 0.02m. Shown in Fig.4 are the results for the vertical displacement responses at point A, B, C, and D over the frequency range of [0-2kHz], which are obtained by using our method and FEM.

It can be seen that those curves coincide with each other very well in low frequency range. In high frequency range, despite the slight discrepancies, it is acceptable to find that the responses from FEM and from DSM have the same trend and lie in the same level. Hence, we can deduce that our method can successfully present good results for vibration responses for three-dimensional beam structures. Using the dynamic stiffness model significantly decreases memory requirement and computational time, and yet retaining high accuracy and high efficiency of the results.
3.2 Vibration transmission: Effect of mass inertia

In this section and below, the vibration isolation effect of the beam raft structure is investigated by analysing the mean square acceleration level differences at the machine mounting, at the beam raft, and at the foundation, respectively.

Fig. 5 shows the vertical acceleration levels at a representative one-third octave frequency 2 kHz with different lumped mass, i.e. 0.02, 0.2, and 2 kg. The other structure and material parameters of the truss raft are the same. The three figures represent three locations of the whole structure. As can be seen from the graphs that the total acceleration level becomes lower with the lumped mass increasing, and the acceleration level difference becomes larger. Increasing the lumped mass can increase the moment of inertia, therefore reducing the vibration responses and obtaining better vibration isolation efficiency.

Table 2: Acceleration level difference with different lumped masses (Units: dB)

<table>
<thead>
<tr>
<th>Mass (kg)</th>
<th>Mounting</th>
<th>Beam raft</th>
<th>Foundation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>116.2</td>
<td>110.2</td>
<td>112.3</td>
</tr>
<tr>
<td>0.2</td>
<td>116.2</td>
<td>109.3</td>
<td>111.4</td>
</tr>
<tr>
<td>2</td>
<td>116.4</td>
<td>106.0</td>
<td>107.3</td>
</tr>
</tbody>
</table>

3.3 Vibration transmission: Effect of connecting angle

In this section, the effect of connecting angle is analysed. Fig. 6 shows calculation curves at one-third octave frequency 2 kHz with different connecting angle, i.e. 60°, 90°, and 120°. Table 3 illustrates detailed values of acceleration level, total acceleration level and acceleration level difference. From Fig. 6 and Table 3, we can find that the three models have the same acceleration level at
the mounting, while as the angle increasing, the acceleration levels at the beam raft and at the foundation increase, thus the acceleration level differences become smaller leading to worse isolation effectiveness. Under unit concentrated force, the whole structure with connecting angle 60° has the best isolation effectiveness 9.0dB and 6.7dB.

![Table 3 Acceleration level difference with different connecting angles](image)

<table>
<thead>
<tr>
<th>Connecting Angle</th>
<th>Mounting</th>
<th>Beam raft</th>
<th>Foundation</th>
</tr>
</thead>
<tbody>
<tr>
<td>60°</td>
<td>116.2</td>
<td>9.0</td>
<td>109.5</td>
</tr>
<tr>
<td>90°</td>
<td>116.2</td>
<td>6.7</td>
<td>111.4</td>
</tr>
<tr>
<td>120°</td>
<td>116.2</td>
<td>4.8</td>
<td>112.4</td>
</tr>
</tbody>
</table>

![Fig. 6 Acceleration level with different connecting angle](image)

**3.4 Vibration transmission: Effect of lower spring arrangement**

Since the upper springs are generally arranged at the feet of isolated machines, their locations are relatively fixed compared to the lower springs. Therefore, in this section, only the arrangement of the lower springs is investigated. Six reasonable forms of distribution are proposed, as shown in Fig.7.

From Fig.8 and Table 4, we can see that the second arrangement has the best effect of vibration isolation. Since the springs are distributed along the center of the lower layer of the raft structure, it is easy to bring the roll vibration to the structure, which should be avoided as far as possible. Therefore, in the design of lower spring distribution, isolator springs should be arranged at the around corners of the truss raft to reduce the roll vibration of the whole structure thus improve the ability of vibration attenuation.

![Fig. 7 Six different types of lower layer spring arrangement](image)

**Table 4 Acceleration level difference with different lower spring arrangements**

<table>
<thead>
<tr>
<th>Location</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mounting</td>
<td>116.2</td>
<td>-</td>
<td>116.6</td>
<td>-</td>
<td>116.5</td>
<td>-</td>
</tr>
<tr>
<td>Beam raft</td>
<td>109.3</td>
<td>6.9</td>
<td>107.9</td>
<td>8.3</td>
<td>113.9</td>
<td>2.7</td>
</tr>
<tr>
<td>Foundation</td>
<td>111.4</td>
<td>4.8</td>
<td>109.8</td>
<td>6.4</td>
<td>116.0</td>
<td>0.6</td>
</tr>
</tbody>
</table>
Fig. 8 Acceleration level with different lower layer spring arrangement.

4. Conclusions

In this paper, the development of dynamic stiffness matrix for the beam element as well as mass, spring-mass joint elements is presented in a straightforward way, and is used for vibration analysis of a periodic built-up beam structure. Accurate results and considerably higher efficiency are obtained, which demonstrates that our proposed DSM approach has great potentials in modelling the dynamics of complex built-up beam structures. Moreover, we have further investigated some factors that influence the vibration isolation effect. The transmission characteristics of other types of beam structure will be studied in our future research.

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REFERENCES


Appendix A:

The dynamic stiffness matrix in Eq. (3) are defined as follows:
The dynamic stiffness matrix in Eq. (4) are defined as follows:

\[
K_w = \begin{bmatrix}
  k_1 & k_2 & k_3 & k_5 \\
  k_2 & k_3 & -k_5 & k_6 \\
  k_4 & -k_5 & k_1 & -k_2 \\
  k_5 & k_6 & -k_2 & k_3 \\
\end{bmatrix}
\]

\[
K = \begin{bmatrix}
  k_7 & g_1 & g_2 & g_4 & g_5 \\
  g_1 & k_1 & k_2 & k_4 & k_5 \\
  k_2 & k_3 & -k_5 & k_6 \\
  g_4 & -k_5 & g_1 & k_1 & -k_2 \\
  g_5 & g_6 & -g_2 & k_3 & k_9 \\
\end{bmatrix}
\]

\[
g_1 = \frac{c_2 \cdot \lambda^3 (\sin \lambda_2 \cdot \cosh \lambda_2 + \cos \lambda_2 \cdot \sinh \lambda_2)}{d_2}
\]

\[
g_2 = \frac{b_2 \cdot \lambda^2 \cdot \sin \lambda_2 \cdot \sinh \lambda_2}{d_2}
\]

\[
g_3 = \frac{a_2 \cdot \lambda \cdot (\sin \lambda_2 \cdot \cosh \lambda_2 - \cos \lambda_2 \cdot \sinh \lambda_2)}{d_2}
\]

\[
g_4 = -\frac{c_2 \cdot \lambda^3 \cdot (\sin \lambda_2 + \sinh \lambda_2)}{d_2}
\]

\[
g_5 = \frac{b_2 \cdot \lambda^2 \cdot (\cosh \lambda_2 - \cos \lambda_2)}{d_2}
\]

\[
g_6 = \frac{a_2 \cdot \lambda \cdot (\sinh \lambda_2 - \sin \lambda_2)}{d_2}
\]

\[
k_1 = \frac{c_1 \cdot \lambda^3 (\sin \lambda_1 \cdot \cosh \lambda_1 + \cos \lambda_1 \cdot \sinh \lambda_1)}{d_1}
\]

\[
k_2 = \frac{b_1 \cdot \lambda^2 \cdot \sin \lambda_1 \cdot \sinh \lambda_1}{d_1}
\]

\[
k_3 = \frac{a_1 \cdot \lambda \cdot (\sin \lambda_1 \cdot \cosh \lambda_1 - \cos \lambda_1 \cdot \sinh \lambda_1)}{d_1}
\]

\[
k_4 = -\frac{c_1 \cdot \lambda^3 \cdot (\sin \lambda_1 + \sinh \lambda_1)}{d_1}
\]

\[
k_5 = \frac{b_1 \cdot \lambda^2 \cdot (\cosh \lambda_1 - \cos \lambda_1)}{d_1}
\]

\[
k_6 = \frac{a_1 \cdot \lambda \cdot (\sinh \lambda_1 - \sin \lambda_1)}{d_1}
\]

\[
\lambda_2 = \frac{\sqrt{\rho A \omega^2 l^4}}{E l}
\]

\[
a_2 = \frac{EI_y}{l}, b_2 = \frac{EI_y}{l^2}, c_2 = \frac{EI_y}{l^3}
\]

\[
d_2 = 1 - \cos \lambda_2 \cdot \cosh \lambda_2
\]

\[
\beta_1 = \omega \sqrt{\frac{p}{E}}, \beta_2 = \omega \sqrt{\frac{p}{G}}
\]

\[
\beta_2 = \omega \sqrt{\frac{p}{G}}, \beta_2 = \omega \sqrt{\frac{p \cdot J_8}{G \cdot J_4}}
\]