ACOUSTICAL SOURCE RECONSTRUCTION FROM NON-SYNCHRONOUS MEASUREMENTS BY GIBBS SAMPLING

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Acoustical source reconstruction from non-synchronous measurements is a powerful method for achieving large arrays and/or high microphone density by scanning the object of interest from a sequential movement of an arbitrary prototype array. It has attracted great interests recently, since it is beyond the fundamental limitation of working frequency that is determined by the size and microphone density of an array. The problem of the non-synchronous measurements of microphone array boiled down to a matrix completion of a block diagonal spectral matrix. In this paper, the problem of non-synchronous measurements has been investigated in the Bayesian formalism. First, a statistical forward model of non-synchronous measurements is constructed; second, the spectral matrix completion is implemented based on the the Gibbs sampling. In the numerical experiments, convergence diagnosis of Markov chain is illustrated through two approaches (i.e. trace plot and ergodic mean plot). In addition, Matrix Completion Errors (MCE) is also discussed with respect to the frequency range and signal-to-noise ratio (SNR).

Keywords: inverse acoustical problem, non-synchronous measurements, Gibbs sampling, Bayesian formalism

1. Introduction

Acoustic imaging aims at obtaining a visualized sound map from the radiating sound of vibrating objects, which has a wide range of applications in source identification, vibration analysis and machine diagnosis\cite{1, 2, 3}. A fundamental limitation of present works is imposed by a geometry of the array. Specifically, the minimum frequency of reconstruction depends on the dimension of the array and the maximum frequency depends on the distance between neighboring microphones \cite{4}. One solution to achieve a large array and/or high microphone density is to scan the object of interest by moving sequentially a prototype array (i.e. non-synchronous measurements). In comparison to a large array and/or high microphone density array that can acquire simultaneously all the information of the spectral matrix, in particular all cross-spectra, non-synchronous measurements can only acquire a block diagonal spectral matrix, while the cross-spectra between the sequential measurement remain unknown due to the missing phase relationships between consecutive positions. Nevertheless, these unknown cross-spectra are necessary for acoustic reconstruction \cite{5}.
Non-synchronous measurements have been investigated in Ref [5], in which the problem is reformu-
lated as a matrix completion problem of a data missing spectral matrix. The Cyclic Projection (CP) [6]
and Fast Iterative Soft Thresholding Algorithm (FISTA) [7] algorithm was proposed correspondingly to
find a full spectral matrix subject to given constraint of hermitian symmetry, measurement fitting, reduced
rank and spatial continuity of the sound field, which is recognized as the non-intrusive method (it does
not reformulate the propagation progress of current reconstruction methods). It is noted that the CP and
FISTA are proposed from a deterministic perspective. In this paper, the problem of the spectral matrix
completion will be discussed in a statistical perspective.

2. A statistical sound propagation model

Acoustic source distribution \( s(r; \zeta), r \in \Gamma \) and elementary events \( \zeta \) in sample space, is reconstructed
at position \( r \) on the source surface \( \Gamma \) at a given frequency \( \omega \) (the \( \omega \) is omitted here for the simplicity). Let \( r_{m,j} \) be the position of the \( m \)-th microphone in the array, \( m = 1, 2, ..., M \), at the \( j \)-th position of the array,
\( j = 1, 2, ..., J \), and \( y(r_{m,j}; \zeta) \) the acoustic pressure measured at \( r_{m,j} \). The measured pressures and the
source field are typically related through an integral equation of the form

\[
y(r_{m,j}; \zeta) = \int_{\Gamma} F(r_{m,j}|r)s(r; \zeta)d\Gamma + n(r_{m,j}; \zeta),
\]

where \( F(r_{m,j}|r) \) stands for the Green function between \( r \) and \( r_{m,j} \) and where \( n(r_{m,j}; \zeta) \) denotes additive
measurement noise which is independent of \( s(r; \zeta) \) in terms of statistics. To discretize this problem, the
sound sources are projected onto a basis of spatial functions \( \phi_l(r)^L \):

\[
s(r; \zeta) = \sum_{l=1}^{L} c_l(\zeta)\phi_l(r),
\]

where coefficients \( c_1(\zeta), ..., c_L(\zeta) \) denote the \( L \) random variables which produce the stochastic fields
\( s(r; \zeta) \). Equation (1) then becomes

\[
y(r_{m,j}; \zeta) = \sum_{l=1}^{L} G_{m,j,l}c_l(\zeta) + n(r_{m,j}; \zeta),
\]

with

\[
G_{m,j,l} = \int_{\Gamma} F(r_{m,j}|r)\phi_l(r)d\Gamma.
\]

Let \( I_j \) be the number of snapshots at position \( j \), \( y_{ij} \in \mathbb{C}^M \) the column vector containing the \( M \) pressures
measured at the \( i \)-th snapshot and \( j \)-th position of the array, \( c_{ij} \in \mathbb{C}^L \) the sources coefficients column
vector whose elements are \( c_{ij1}, ..., c_{ijL} \), \( n_{ij} \in \mathbb{C}^M \) the column vector of additive noise and \( G_{j} \in \mathbb{C}^{M \times L} \)
the transfer matrix at configuration \( j \). Equation (3) can be reformulated as a more compact form:

\[
y_{ij} = G_{j}c_{ij} + n_{ij}, j = 1, ..., J, i = 1, ..., I_j.
\]

It is further assumed that the source coefficients are generated from the stochastic model

\[
c_{ij} = \Lambda \epsilon_{ij}
\]

with \( \Lambda \in \mathbb{C}^{L \times K} \), where the "virtual sources" \( \epsilon_{ij} \in \mathbb{C}^K \) are mutually uncorrelated and pertaining in a
space of reduced dimension \( K \leq L \) as compared to that of \( c_{ij} \). Now, equation (6) is further simplified into
the following form

\[
y_{ij} = G_{j}\Lambda \epsilon_{ij} + n_{ij}, j = 1, ..., J, i = 1, ..., I_j.
\]
3. **Gibbs sampling in the Bayesian formalism**

3.1 **Problem statement**

The main purpose of inverse problem is to recover the spectral matrix which is addressed by estimating $\Lambda$, the variances of virtual sources $\alpha^2$ (i.e. $Var(\epsilon_{k,i,j})$) and the variances of noise $\beta^2$ (i.e. $Var(n_{m,i,j})$) while only $y_{ij}$ is measured. Let $S_{yy}^j$ be the partial cross-spectral matrices (CSM) of the measurements at position $j$, and it can be written as follows:

$$S_{yy}^j = 1 \frac{1}{I_j} \sum_{i=1}^{I_j} y_{ij} y_{ij}^H, j = 1, ..., J.$$  \hfill (8)

Then, it results from Eq. 8 that

$$S_{yy}^j = G_j \Lambda \text{Diag}(\alpha^2) \Lambda H G_j^H + \text{Diag}(\beta^2), j = 1, ..., J,$$  \hfill (9)

where $\text{Diag}(\alpha^2)$ stands for diagonal matrix composed with an element $\alpha^2$. The full spectral matrix then can be completed as $S_{yy}^{\text{com}} = G \Lambda \text{Diag}(\alpha^2) \Lambda H G^H$ when the $\Lambda$ and $\alpha^2$ is solved. This will be discussed in the next subsection.

3.2 **Gibbs sampling for the spectral matrix completion**

This section addresses the inference based on the Gibbs sampling. In this framework, all variables are modeled as random variables. The following hierarchial model is assumed for the prior probability density functions (PDF):

- $n_{m,i,j} \sim \mathcal{N}_c(0, \beta^2)$,
- $\Lambda_{kl} \sim \mathcal{N}_c(0, K^{-1})$ such that, a prior, $E(\Lambda \Lambda^H) = I_N$,
- $\epsilon_{k,i,j} \sim \mathcal{N}_c(0, \alpha^2)$ such that, a prior, $E(\epsilon_{k,i,j} \epsilon_{k,i,j}^H) = \text{Diag}(\alpha^2)$,
- $\beta^2 \sim \mathcal{IG}(a_\beta, b_\beta)$,
- $\alpha^2 \sim \mathcal{IG}(a_\alpha, b_\alpha)$,

where $\mathcal{N}_c$ stands for the complex Gaussian PDF and $\mathcal{IG}$ for the inverse gamma PDF.

The principle of the Gibbs sampling is to iterate draws in the posterior PDFs of all variables of interest (i.e. conditioned on the observation of the data). One difficulty in this formulation is that the CMS $S_{yy}^j$ leads to complicated PDFs to sample from. In order to circumvent this difficulty, it is proposed to temporarily introduce the data $y_{ij}$, as if they were observed. The notation $[x]$ is used to denote the PDF of $x$. Then, the Gibbs sampling consists in iterating the following draws:

- $\epsilon_{ij}^{[n]} \leftarrow [\epsilon_{ij} | \{y_{ij}\}, \Lambda^{[n-1]}, \beta^{2,[n-1]}, \alpha^{2,[n-1]}]$,
- $\Lambda^{[n]} \leftarrow [\Lambda | \{y_{ij}\}, \{\epsilon_{ij}^{[n]}\}, \beta^{2,[n-1]}, \alpha^{2,[n-1]}]$,
- $\beta^{2,[n]} \leftarrow [\beta^{2} | \{y_{ij}\}, \{\epsilon_{ij}^{[n]}\}, \{\Lambda^{[n]}\}, \alpha^{2,[n-1]}]$,
- $\alpha^{2,[n]} \leftarrow [\alpha^{2} | \{y_{ij}\}, \{\epsilon_{ij}^{[n]}\}, \{\Lambda^{[n]}\}, \beta^{2,[n]}]$.

Let $\neg$ denote excluding the variable (e.g. the notation $\neg \epsilon_{ij}$ contains all variables of the inverse problem except $\epsilon_{ij}$ itself.). The expressions of the above posterior PDFs of parameters are given hereafter.

[1] The posterior PDF on $\epsilon_{ij}$ is

$$[\epsilon_{ij} | \neg \epsilon_{ij}] \propto [y_{ij} | \epsilon_{ij}, \Lambda, \beta^2, \alpha^2][\epsilon_{ij}]$$

$$\propto \mathcal{N}_c(y_{ij} - G_j \Lambda \epsilon_{ij}, 0, \text{Diag}(\beta^2)) \mathcal{N}_c(0, \text{Diag}(\alpha^2)) \mathcal{N}_c(\mu_{ij}, \Omega_j)$$  \hfill (10)
with

$$
\begin{align*}
\mu_{ij} &= B_j y_{ij} \\
B_j &= \Omega_j \Lambda \Lambda^H G_j^H \text{Diag}(\beta^{-2}) \\
\Omega_j^{-1} &= \Lambda^H G_j^H \text{Diag}(\beta^{-2}) G_j \Lambda + \text{Diag}(\alpha^{-2}).
\end{align*}
$$

(11)

Let’s introduce $\lambda = \text{vec}\{\Lambda\}$ such that $y_{ij} = (\epsilon_{ij}^T \otimes G_j) \lambda + n_{ij}$ (the notation $\otimes$ denotes Kronecker product).

[2] The posterior PDF on $\lambda$ is

$$
[\lambda|\neg \lambda] \propto \prod_{ij} \mathcal{N}_c(y_{ij} - (\epsilon_{ij}^T \otimes G_j) \lambda | 0, \text{Diag}(\beta^2)) \mathcal{N}_c(0, K_{N\times K}^{-1})
$$

(12)

and

$$
\begin{align*}
\mu_{\lambda} &\simeq \sum_j I_j \text{vec}\{G_j^H \text{Diag}(\beta^{-2}) S_{yy}^j B_j^H\} \\
\Omega_{\lambda}^{-1} &\simeq \sum_j (S_{\epsilon\epsilon}^j)^* \otimes (G_j^H \text{Diag}(\beta^{-2}) G_j) + K I_{N\times K}
\end{align*}
$$

(13)

with

$$
\begin{align*}
S_{\epsilon\epsilon}^j &= I_j B_j S_{yy}^j B_j^H + \Omega_{\lambda}^{1/2} W_{\epsilon\epsilon}^j \Omega_{\lambda}^{1/2} \\
W_{\epsilon\epsilon}^j &= I_j \text{vec}\{G_j^H \text{Diag}(\beta^{-2}) S_{yy}^j B_j^H\}
\end{align*}
$$

(14)

where $\mathcal{W}_{\epsilon\epsilon}^j$ is a complex Wishart matrix with $I_j$ degrees of freedom.

[3] The posterior on $\beta^2$ is

$$
[\beta^2|\neg \beta^2] \propto \prod_{ij} \mathcal{N}_c(y_{ij} - G_j \Lambda \epsilon_{ij}|0, \text{Diag}(\beta^2)) \mathcal{IG}(a_\beta, b_\beta)
$$

(15)

with

$$
\begin{align*}
E_y^j &\simeq I_j P_j S_{yy}^j P_j^H + G_j \Lambda \Omega_j \Omega_j^{1/2} W_{\epsilon\epsilon}^j \Omega_j^{1/2} \Lambda^H G_j^H \\
P_j &= I_M - G_j \Lambda B_j \text{ and } z_{ij} \sim \mathcal{N}_c(0, I_k).
\end{align*}
$$

(16)
The posterior on $\alpha^2$ is

$$[\alpha^2|\neg\alpha^2] \propto \prod_{ij} [y_{ij}|\epsilon_{ij}, \lambda, \beta^2, \alpha^2][\alpha^2][\epsilon|\alpha^2]$$

$$\propto \prod_{ij} N_c(y_{ij} - G_j \Lambda \epsilon_{ij}|0, Diag(\beta^2)) \mathcal{IG}(a_\alpha, b_\alpha) \prod_{ij} N_c(0, \alpha^2)$$

$$\propto \mathcal{IG}(a_\alpha + K \sum_j I_j, b_\alpha + \text{trace}(\sum_j I_j S_{\epsilon \epsilon})).$$

(17)

4. Numerical experiments

Figure 1: Trace plots at 2000 Hz with SNR = 20 dB: (a) $\alpha^2$, (b) $\beta^2$, (c) the real part of elements in the matrix $\Lambda$, (d) the imaginary part of elements in the matrix $\Lambda$.

In this section, the proposed method is tested on synthetic data under three simulations. The first two numerical experiments are implemented to validate convergence of Markov chain in the Gibbs sampling. In addition, the MCE is used to validate the performance of the method in the third numerical
Figure 2: Ergodic mean plots at 2000 Hz with SNR = 20 dB about different parameters: (a) $\alpha^2$, (b) $\beta^2$, (c) the real part of elements in the matrix $\Lambda$, (d) the imaginary part of elements in the matrix $\Lambda$. 
experiments. Generally, the smaller MCE means a better performance of the method. The acoustical field is generated by three points with same magnitudes at different location (i.e. \((0.2701 \text{ m}, -0.0084 \text{ m}), (-0.1613 \text{ m}, 0.2348 \text{ m}), (0.0641 \text{ m}, 0.1573 \text{ m})\)) which build up a homogeneous infinite acoustic domain. In the sound field, the air mass density is 1.29 kg/m\(^3\) and the sound velocity is 341 m/s. To compare the MCE of source reconstruction, the source plane \((1 \text{ m} \times 1 \text{ m})\) is discretized equally into a \(41 \times 41\) grid where mesh grid is 0.025m. The measurement plane is located 0.05 m away from the sound source plane where the microphones are uniformly distributed with spacing 0.025 m in a square array \((1 \text{ m} \times 1 \text{ m})\). A prototype array is translated 9 times sequentially at positions \((-0.16, -0.16), (-0.16, 0.08), (-0.16, 0.16), (0.08, -0.16), (0.08, 0.08), (0.16, -0.16), (0.16, 0.08)\) and \((0.16, 0.16)\) (meters). The parameters of Gibbs sampling within the Bayesian formalism for the inverse acoustics are listed here: number of configurations \(J = 9\), number of sensors in the antenna \(M = 25\) and number of snapshots \(I_j = 100\). The spectral matrix completion errors is calculated as

\[
MCE = \frac{\|S_{yy}^{sim} - S_{yy}^{com}\|_F}{\|S_{yy}^{sim}\|_F},
\]

where \(S_{yy}^{sim}\) is the true spectral matrix and \(S_{yy}^{com}\) is the completed one.

The convergence of Markov chain in the Gibbs sampling on critical parameters value is illustrated in Figs. 1 and 2. Figure 1 illustrates the trace plots of important parameters (i.e. \(\alpha^2, \beta^2, \Lambda\)) at 2000 Hz with SNR = 20 dB, which prove the convergence of Markov chain. Figure 2 illustrates the ergodic mean plot of estimated parameters under the same frequency and SNR with the trace plot. Figure 3 shows the completion errors at SNR = 10 dB or 20 dB with respect to the frequency range \((300 \text{ Hz} \sim 6000 \text{ Hz})\), where the number of iterations in Gibbs sampling is 1000 and the burn-in period \([8]\) (not to be considered for the computation of target distribution) is 100. It is noted that the limited high frequency of the using

![Figure 3: MCE results are shown: the case with SNR = 10 dB by circles; with SNR = 20 dB by asterisks.](image-url)
array is 1700 Hz, the MCE until 6000 Hz has been illustrated in the figure which extended the working frequency range by using a single array.

5. Conclusion

In this paper, the non-synchronous measurements is reformulated in the Bayesian framework, and the sound sources are reconstructed by the Gibbs sampling. It is addressed by Gibbs sampling from the full conditional distribution of posterior PDFs of parameters in the inverse acoustics. A full spectral matrix can also be correspondingly completed, which is considered as a figure of merit to evaluate the proposed method. Eventually, the simulation results show that the potential of Gibbs sampling can be further applied in the acoustics inverse problem.

6. Acknowledgement

This work was supported by National Natural Science Foundation of China (Grant No. 11704248) and Open Fund of the Key Laboratory of Aerodynamic Noise Control (ANCL20180302).

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