There has been a long debate whether wall vibrations can affect the sound produced by brass wind instruments. During the last decade unambiguous evidence of the effect of wall vibrations has been provided by experimental measurements. However the underlying mechanism was still unclear, although several explanations had been posited. This led towards physical modelling approaches that aimed at simulating the observed effects, while shedding some light on the way the vibrating walls could significantly alter the sound generated by the instrument. It turns out that axial vibrations of the instrument bore are responsible for the observed effects. Namely, the wall vibrations due to the acoustic pressure dynamically alter the internal bore radius and couple to the air column inside the instrument. It has been shown that this is a broadband effect that may significantly affect the acoustic impedance of the instrument over a considerable frequency range. Numerical simulations using the finite element method verified this behaviour. Subsequently, simplified models were constructed in order to yield efficient simulation tools that are able to capture this effect. By representing the instrument using a series of masses and springs, it has been possible to reproduce the results obtained with the finite element method, with the exemption of one mode of vibration that is related to shear stresses and strains. This paper shows how to incorporate this mode to the aforementioned mass-spring model, which may be coupled to a wave-propagation model for vibroacoustic simulations. The resulting numerical scheme remains computationally efficient in comparison to the finite element model (approximately 100 times faster) and may be used in series of simulations, for example for bore optimisation, sound synthesis and sound analysis of brass wind instruments.

Keywords: brass wind instruments, wall vibrations, physical modelling

Introduction

The effect of wall vibrations on the sound of brass wind instruments has been thoroughly investigated in the past decades [1] [2] [3] [4] [5]. This has led to a debate whether different wall materials may have a significant influence on the radiated sound [6]. Recent theoretical and experimental findings suggest that vibrations along the axis of the instrument bore (hereafter referred to as axisymmetric vibrations) may
result in a broadband effect on the input impedance and transfer function of the instrument \cite{7, 8}, and hence significantly affect sound radiation.

In order to capture such effects in a numerically efficient manner (e.g. for purposes of bore optimisation or sound synthesis) a simplified mass-spring model has been proposed, able to simulate the oscillations of the instrument walls \cite{9}. This model can predict the axisymmetric resonances of a straight bell in the playing range of brass instruments, apart from one modal shape that involves a node very close to the rim of the bell. This particular modal shape is predicted both by a finite element model (FEM) \cite{9} and using laser interferometry \cite{10}. This study shows how the mass-spring model may be extended in order to capture this mode of vibration.

The next section gives an outline of the mass spring model, discussing its efficiency and limitations. Section 3 describes how shear stiffness and strain may be added to this model and compares the obtained results with FEM and Section 4 discusses the results of this study.

**Mass spring model**

Towards formulating a mass-spring model that simulates the wall vibrations of a brass instrument a simplified geometry is first assumed. Only a straight bell is simulated, assuming a perfectly axisymmetric geometry (though real, handcrafted bells may show deviations from such a shape) with a constant thickness. Since it is significant to the modal shapes, a rim wire is also included in the model. The distribution of mass and stiffness along the bell is discretised using a finite number of masses and springs, as illustrated in Figure 1 (left). According to the observations mentioned in the Introduction, only axisymmetric effects are considered, neglecting any stress or strain along the circumferential direction. Calculating all the forces acting on this discretised system (including external pressure forces and internal stresses) leads to a system of partial differential equations that may be numerically solved.

Under the axisymmetry assumption, each lumped mass corresponds to the mass of the equivalent circular brass segment and is given by

\[
m_i = \frac{\pi \rho h (2r_i \beta + \beta^2)}{\cos \theta_i},
\]

where \(\rho\) is the density of brass, \(m_i\) is the mass of segment \(i\), \(r_i\) its internal radius, \(\beta\) its thickness and \(h\)
the axial distance between the masses (i.e. the grid size). The length of each segment at rest is given by \( L_i = \frac{h}{\cos \theta_i} \), were \( \theta \) is the flare angle.

During performance, the internal pressure \( p \) that builds up results in a force \( F_p \) that acts on the walls of each segment and is given by

\[
F_p(i) = 2\pi r_i p_i h / \cos \theta_i.
\]  

This force is always perpendicular to the wall. Therefore, the radial and axial components of the force due to internal pressure can be calculated as

\[
F_r(i) = 2\pi r_i p_i h
\]
\[
F_x(i) = -2\pi r_i p_i h \tan \theta_i.
\]

The spring constants of the springs between the masses may be calculated using Hooke’s law \([6]\) and are given by

\[
c_i = \frac{2\pi r_i \beta E}{h / \cos \theta_i} = \frac{2\pi r_i \beta E \cos \theta_i}{h},
\]

where \( E \) is the Young’s modulus. The radial spring constants can be calculated using the definition of a spring constant as \( k_i = F_r(i) / s_i \), where \( s_i = p_i r_i^2 / (E \beta \cos^3 \theta_i) \) is the amplitude of the radial wall displacement and \( F_r(i) \) the radial pressure force \([6]\). Therefore, the radial spring constants are given by

\[
k_i = 2\pi E h \cos^3 \theta_i / r_i.
\]

At the rim of the bell, the brass is folded around a rim wire thus increasing the mass of the last segment. This has a significant effect on the structural resonances of the bell \([9]\). Finally, the Poisson effect needs to be considered in the model. A strain in one direction results in a stretch in another direction, which is described using the Poisson ratio of the material. However, in this case, since the displacements in the axial direction are much greater than those in the radial direction only processes in which the radial displacement is affected by the axial displacement are considered. This allows to decouple the equation of motion for each direction (axial and radial) resulting in the following system of equations in the frequency domain

\[
\begin{align*}
\left\{ \begin{array}{l}
c_i X_{i-1} + (m_i \omega^2 - c_i - c_{i+1}) X_i + c_{i+1} X_{i+1} + F_x(i) = 0 \\
c_i Y_{i-1} + (m_i \omega^2 - c_i - c_{i+1} - k_i) Y_i + c_{i+1} Y_{i+1} + F_r(i) = 0,
\end{array} \right.
\end{align*}
\]

where \( X_i \) and \( Y_i \) correspond to the complex amplitude of the axial and radial displacement of mass \( m_i \) and \( \omega \) is the angular frequency. The total displacement in the radial direction can be subsequently calculated by adding the contribution from the axial displacement,

\[
Y_{tot} = Y_i + Y_{X_i} = Y_i - r_i \nu \frac{X_{i+1} - X_{i-1}}{2h},
\]

where \( \nu \) is Poisson’s ratio. Solving equations (7) and (8) for each frequency makes it possible to determine the displacement at any point on the wall.

In order to validate this simplified model, a comparison is carried out with a benchmark finite element model \([11]\). It turns out that the mass-spring model can simulate most of the axisymmetric dynamics of the trumpet bell. Figure 2 shows the displacement amplitude along the bell predicted by both models. The
Figure 2: Rim displacement over frequency simulated using the mass-spring model (top) and using the finite element method (bottom).

The main difference between the two models is the fact that an additional mode appears in the finite element simulations that is not predicted by the mass-spring model. This mode is associated with a deflection shape with a node very close to the rim of the bell (see Figure 1 right). In order for the simplified mass-spring model to be able to capture this mode, shear stiffness and strain need to be taken into account, as explained in the next section.

A further difference concerns the position of the peaks in the displacement curve. Both the first and fourth peaks in the finite element model are located at higher frequencies corresponding to the respective first and third peak of the mass-spring model (note that the third peak in the FEM plot has no counterpart in the mass-spring model). This shows that the finite element model is more stiff, possibly due to a more accurate representation of the direction of the internal forces.

**Shear stiffness and strain**

Approximating the continuous bore profile using a discrete series of masses and springs allows to arrive at a numerical expression of the wall vibrations. The number of masses (and accordingly the distance between them) is chosen such that the algorithm is both accurate and efficient. In practice a distance of 2 mm has been shown to be sufficient. However, when considering the vibrations at the rim of the bell, the rim-wire itself has a diameter of 3 mm. Hence, it is an oversimplification to simply add its mass to the end of the mass-spring system. Furthermore, shear stresses and strains have been previously ignored due to the fact that the bell walls are very thin. This assumption has been validated by simulations of trumpet bells in the absence of a rim wire [7]. However, at the rim region their effect becomes significant due to the additional thickness introduced by the rim wire. In order to consider the above effects, an extension to the mass-spring model is proposed, as depicted in Figure 3. The rim mass is now located at the centre of gravity of the rim, and additional springs are added in the area covered by the rim wire, in order to account for the shear stresses in that region. These could not be captured...
Figure 3: Mass-spring distribution for a model including both shear and axial springs in the vicinity of the rim wire.

by either the axial springs that lie along the bore profile, nor the radial springs that only account for circumferential strain.

The spring constants of these additional shear springs may be also calculated by Hooke’s law and are given by

$$\kappa_i = \frac{2\pi r_i \beta G}{\lambda_i},$$

(9)

where $G$ is the shear modulus of brass [12] and $\lambda$ the distance of each mass from $m_n$. Equations (7) are modified accordingly as follows:

$$c_i X_{i-1} + (m_i \omega^2 - c_i - c_{i+1} - \kappa_i) X_i + c_{i+1} X_{i+1} + \kappa_i X_n + F_x(i) = 0$$

$$c_i Y_{i-1} + (m_i \omega^2 - c_i - c_{i+1} - k_i - \kappa_i) Y_i + c_{i+1} Y_{i+1} + \kappa_i Y_n + F_r(i) = 0.$$  

(10)

In terms of the numerical solution, the main difference of the updated equations (10) compared to the previously used equations (7) is that the resulting system is not tridiagonal any more, due to the connection of a number of masses near the rim to the rim mass. Nevertheless, the system matrix remains sparse leading to an efficient numerical algorithm. In fact, the presented model outperforms the benchmark finite element model by a factor of 100, for the same grid size.

**Results**

It can be observed that the addition of the shear springs to the mass-spring model resulted in an additional axial resonance of the trumpet bell. Figure 4 shows a comparison with the finite element model. This resonance lies between the second and the fourth resonance, as expected, and could not be obtained by using the previous version of the mass-spring model. Furthermore, the presence of this mode results in a shift of the next mode towards higher frequencies; this fourth mode now lies closer to the one
observed by FEM. Overall, the qualitative agreement between the simplified and efficient mass-spring model and the elaborate and accurate finite element model has been improved by the addition of the shear springs.

**Conclusions**

In this paper a refined mass-spring model has been presented that is able to simulate trumpet wall vibrations in qualitative agreement with a benchmark finite element model. It is based on a previously presented mass-spring model that was neglecting any shear stresses of the instrument walls. The addition of shear springs in the vicinity of the rim-wire improved the results in terms of capturing axisymmetric modes of vibrations as identified by both finite element simulations and optical measurements. Despite this qualitative agreement, the resonance frequencies do not perfectly match for all modes. To this end, further refinement of the mass-spring model is required using data obtained by both numerical simulations and experimental measurements.

**REFERENCES**


