NON-LINEAR VIBRATIONS NEAR RESONANCE OF LAMINATED COMPOSITE BEAMS

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The non-linear vibrations of simply-supported laminated composite beams are investigated by using the normal modes method and the method of multiple scales. Closed-form analytical solution is determined and compared with finite element solution. For the closed-form solution, both the cylindrical bending theory and one-dimensional laminated beam theory are considered. For both the cases, the frequency-response curves and backbone curves are obtained for near-resonant oscillations of the composite beam. The excitation frequency and amplitude at which the beam shows significant non-linearity and the dependence of non-linearity on excitation amplitude are investigated. The relative accuracy of solutions obtained based on the cylindrical bending theory and one-dimensional laminated beam theory in describing the non-linear vibrations during passage through resonance is evaluated. A parametric study is performed in order to investigate the dependence of non-linearity on the slenderness ratio and the aspect ratio of the beam.

Keywords: Geometric non-linearity, non-linear vibrations, multiple scale method, laminated composite beams

1. INTRODUCTION

The strength and stiffness of composite laminated beam depend significantly upon the arrangement of various laminae through the thickness, fiber orientation in each lamina, and the material properties of the laminae. The possibility to optimize the laminate strength increases by varying the lay-up configuration of the laminated beams. In the literature most of the works were done on the linear vibrations of the composite laminated beams. Non-linear aspects still remain to be investigated.

The steady-state harmonic response of a beam is linear when the excitation frequency and amplitude are small. However, when the excitation frequency approaches the natural frequencies of the beam, the beam starts exhibiting significant non-linear behaviour. The non-linearities of the beam develop through
diverse sources such as geometrical, material or inertial nonlinearities. Non-linear mid-plane stretching or large curvatures are the major sources of the deformation.

When the simply-supported beam is subjected to transverse vibrations, geometric non-linearity is developed due to the mid-plane stretching. The non-linear free vibrations of a beam with slowly varying properties along its length were studied by Nayfeh [1] using the multiple scale perturbation method. A numerical-perturbation method to determine the forced response of a non-uniform beam for the structural vibrations was presented by Nayfeh et al [2]. Rao et al. [3] presented the ratios of non-linear frequency to linear frequency for the fundamental mode for various values of slenderness ratios and central amplitude ratios. Heyliger and Reddy [4] investigated the influence of in-plane inertia and slenderness ratio on the non-linear frequency of beams. Nonlinear free vibration of laminated composite beam was studied by Kapania and Raciti [5] to investigate the large-amplitude vibrations. They developed a composite beam element based on first-order shear-deformation theory. The dynamic finite element equations were reduced to a single second-order ordinary nonlinear differential equation using the linear mode shape of vibration, which is solved using the perturbation method. Large-amplitude free vibrations of unsymmetrically laminated beam using von-Karman large deflection theory was investigated by Singh et al. [6] using one-dimensional finite elements based on classical lamination theory, first-order shear-deformation theory, and higher-order shear-deformation theory.

The present study is motivated by the lack of a comparative study on non-linear vibrations of laminated beams of different lay-up configurations based on both the cylindrical bending theory and the one-dimensional laminated beam theory. The objective of the present paper is to present a formulation for the motions of composite laminated beams to determine explicitly the dependence of response non-linearity on various structural and loading parameters including aspect ratio, slenderness ratio, excitation frequency, and excitation amplitude. The parameter values at which the composite beam starts showing significant non-linearity under harmonic excitation are studied. In the formulation, the oscillations of the beam are represented as combination of free vibration linear modes and consequently, a set of second-order coupled equations is obtained and they are solved using the multiple scale perturbation method.

2. HARMONIC FORCED VIBRATION OF THE COMPOSITE BEAM

As shown in Fig. 1 a simply-supported beam subjected to a uniformly distributed harmonic force is considered.

2.1 LINEAR VIBRATION OF THE BEAM

The equation of motion for the forced lateral vibration of a uniform beam in z-direction can be written as

\[
EI \frac{\partial^4 Z(x, t)}{\partial x^4} + \rho A \frac{\partial^2 Z(x, t)}{\partial t^2} = f_z(x, t)
\]  

(1)
where $E$ is modulus of the beam, $I$ represents moment of inertia about $y$-axis, $\rho$ is the density of the beam, $A$ is cross-section of the beam, $Z$ is the direction of vibrations. For composite beam, EI will be written in terms of laminate stiffnesses as explained in a later section. The forced vibration response of the beam can be evaluated using the normal mode method and correspondingly, the deflection of the beam can be represented as [7]

$$Z(x, t) = \sum_{n=1}^{N} Z_n(t) \phi_n(x)$$

(2)

where $\phi_n(x)$ represents the linear undamped natural modes of the simply-supported beam and $Z_n(t)$ represents the generalized coordinate in the $n^{th}$ mode.

By substituting the assumed harmonic response Eq. (2) into the harmonic forced vibration Eq. (1), one can obtain

$$EI \sum_{n=1}^{N} \frac{\partial^2 \phi_n(x)}{\partial x^2} Z_n(t) + \rho A \sum_{n=1}^{N} \phi_n(x) \frac{\partial^2 Z_n(t)}{\partial t^2} = f_Z(x, t)$$

(3)

Linear undamped natural modes of the simply-supported beam and harmonic excitation force can respectively be given as

$$\phi_n(x) = \sin\left(\frac{n\pi x}{l}\right)$$

(4)

$$f_Z(x, t) = K \cos(\omega t)$$

(5)

The steady-state response of the simply-supported beam subjected to harmonic loading can be obtained as [7]

$$Z_n(x, t) = \frac{2K}{ml} \frac{\sin\left(\frac{n\pi x}{l}\right)}{\omega_n^2 - \omega^2} \cos(\omega t)$$

(6)

where $m$ represents mass per unit length, $\omega_n$ represents the natural frequency ($\omega_n = n \pi \sqrt{EI/\rho Al^4}$), $\omega$ represents the excitation frequency and $K$ is the excitation amplitude in $z$-direction.

### 2.2 NON-LINEAR VIBRATION OF THE BEAM

In this section, the non-linear vibration response of the continuous beam system is studied and an approximate analytical approach is used to determine the non-linear response of the beam. Following assumptions are considered: (i) structure remains planar, (ii) non-rotating beam, (iii) linear stress-strain relationship, (iv) radius of gyration of the beam cross-section must be small compared to the wavelength of the transverse vibrations, (v) neglect transverse shear and rotary inertia effects.

Using above assumptions, from the equation of motion of the beam, one can arrive at the second-order coupled differential equation for the first mode of vibration as [8]:

$$Z_{tt} + n^4 \pi^4 \left(\frac{EI}{ml^4}\right) Z = \epsilon\left(-\frac{1}{4} n^4 \pi^4 \left(\frac{EI}{ml^4}\right) Z^3 - \mu Z_t + \frac{2k \cos(\Omega t)}{ml}\right)$$

(7)

where,

$$f_Z(x, t) = \epsilon k \cos(\Omega t)$$

(8)

and $\epsilon$ represents a small bookkeeping parameter, which tracks different orders of approximation. Here the excitation force is represented in such a way ($i.e., K = \epsilon k$) that it appears when non-linearity appears. For composite beam, EI will be written in terms of laminate stiffnesses as explained in a later section.

### 2.3 MULTIPLE SCALE PERTURBATION SOLUTION

The multiple time-scales method [8] is used to investigate the frequency response and steady-state response of the simply-supported beam.
Introducing the detuning parameter (i.e., $\sigma$) in excitation frequency, which expresses the nearness to the resonance,

$$\Omega = \omega_n \pm \sigma \epsilon$$  \hspace{1cm} (9)

and using the method of multiple scales perturbation, the oscillatory response can be represented as

$$Z(t; \epsilon) = Z_0(T_0, T_1, T_2) + \epsilon Z_0(T_0, T_1, T_2) + \cdots$$  \hspace{1cm} (10)

where $T_0, T_1$ and $T_2$ represent the fast and slow time scales.

Substituting Eq. (9) and Eq. (10) into Eq. (7) and by expanding the derivatives and equating coefficients of same power of $\epsilon$ leads to the following equations.

$$O(\epsilon^0) : \quad D_0^2 Z_0 + \omega_0^2 Z_0 = 0,$$  \hspace{1cm} (11)

$$O(\epsilon^1) : \quad D_1^2 Z_1 + \omega_0^2 Z_1 = -2D_0 D_1 Z_0 - \frac{1}{4} \omega_0^2 Z_0^2 - \mu Z_0 + \frac{2k \cos(\Omega t)}{ml}$$  \hspace{1cm} (12)

where,

$$D_n = \frac{\partial}{\partial T_n}$$  \hspace{1cm} (13)

Solution of the Eq. (11) can be written as

$$Z_0 = A (T_1, T_2) e^{i\omega_0 T_0} + \bar{A} (T_1, T_2) e^{-i\omega_0 T_0}$$  \hspace{1cm} (14)

Substituting Eq. (14) into the right hand side of the Eq. (12) and eliminating the secular terms, one gets

$$2i (D_1 (A) + \mu A) + \frac{3}{4} \omega A^2 \bar{A} - \frac{k}{ml \omega_n} e(i\epsilon T_0) = 0$$  \hspace{1cm} (15)

Assuming polar form of $A$,

$$A = \frac{1}{2} a (T_1) e^{i\beta(T_1)}$$  \hspace{1cm} (16)

and introducing an autonomous system

$$\gamma = \sigma T_1 - \beta$$  \hspace{1cm} (17)

Substituting Eq. (16) into Eq. (15) and solving for real and imaginary terms separately one can obtain the frequency- response equation as

$$\sigma = \frac{3}{32} \omega_n a^2 \pm \sqrt{\left( \frac{\pi^2 k}{ml \omega_n a^2} \right)^2 - \left( \frac{\mu}{ml} \right)^2}$$  \hspace{1cm} (18)

where the first term on the right hand side of Eq. (18) represents non-linear effect and second term represents linear effect.

Frequency-response equation represents a case equivalent to the non-linear hard spring. Consequently, the approximation of the steady-state response of the beam for the first mode can be written as

$$Z(x, t) = a \cos(\Omega t - \gamma) \sin(\pi x) + O(\epsilon)$$  \hspace{1cm} (19)

### 3. ANALYSIS OF LAMINATED BEAMS

In one-dimensional beam analysis, the length of the beam is considered to be much greater than its width. Such a case is similar to plain stress theory in elasticity. Symmetric laminates are studied for which there is no stretching-bending coupling.

In the case of pure bending of a symmetric laminate the constitutive equation can be written as [9]

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix}$$  \hspace{1cm} (20)

where the coefficients of the [D] matrix are determined using the ply and laminate properties and corresponding expressions are available in Ref. [9]. The matrix $\{k\}$ is the curvature matrix of the plate subjected to bending and twisting.

Equation (20) can be written in the inverted form as

$$\begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} = \begin{bmatrix} D^*_{11} & D^*_{12} & D^*_{16} \\ D^*_{12} & D^*_{22} & D^*_{26} \\ D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}$$  \hspace{1cm} (21)
where \( D_{ij}^* \) are the elements of the inverse matrix of \([D_{ij}]\) and for the case of symmetric laminate \( D_{16} = D_{26} = 0 \). In the case of bending along \( x \) axis the moment \( M_y \) and \( M_{xy} \) are assumed to be zero.

\[
\begin{align*}
D_{11}^* &= \frac{1}{\Delta} (D_{22}D_{66} - D_{26}^2) \\
D_{12}^* &= \frac{1}{\Delta} (D_{16}D_{26} - D_{12}D_{66}) \\
D_{22}^* &= \frac{1}{\Delta} (D_{11}D_{66} - D_{16}^2) \\
\Delta &= D_{11}D_{22} + 2D_{12}D_{16}D_{26} - D_{11}D_{26}^2 - D_{22}D_{16}^2 - D_{66}D_{12}^2
\end{align*}
\]  

(22)

Using Eq. (22), (23), (24) and (25) one can obtain \( D_{11} \) as:

\[
D_{11} = \frac{1}{D_{11}^* \left( 1 - \left( \frac{D_{12}^*}{D_{11}^*} \right)^2 \left( \frac{D_{11}^*}{D_{22}^*} \right) \right)}
\]

(26)

Using above assumption and Equation (21), one can conclude that:

\[
k_x = -\frac{\partial^2 Z_0}{\partial x^2} = D_{11}^* M_x
\]

(27)

Since beam has high length-to-width ratio, the deflection is assumed as function of \( x \) only

\[
Z_0 = Z_0(x)
\]

(28)

Combining Eq. (27) and Eq. (28), one can obtain

\[
\frac{\partial^2 Z_0}{\partial x^2} = -\frac{M}{E_b^b I}
\]

(29)

where,

\[
D_{11}^* = \frac{12}{E_b^b h^3}, M = b M_x, I = \frac{b h^4}{12}
\]

(30,31,32)

where, \( E_b^b \) is the effective bending modulus of the beam in \( x \)-direction and \( I \) is the moment of inertia in \( y \)-direction. Flexural vibration frequency of a laminated strip by this 1D beam theory can be written as [9]

\[
\omega_{nba} = n^2 \pi^2 \sqrt{\frac{1}{D_{11}^* \rho l^4}}
\]

(33)

where \( D_{11}^* \) can be obtained from Eq. (30).

In cylindrical bending theory, the laminate is considered to have very high length-to-width ratio such that plate deformation can be considered as independent of the coordinate along the length. Such case is similar to plane strain theory of elasticity. The beam has a large length in \( y \)-direction and uniformly supported along the edges \( x = 0, x = l \). Flexural vibration frequency of a laminated strip by cylindrical bending can be written as [9]

\[
\omega_{ncb} = n^2 \pi^2 \sqrt{\frac{D_{11}}{\rho l^4}}
\]

(34)

where \( D_{11} \) can be obtained from Classical Laminated Plate Theory [9].

The natural frequencies obtained based on both the theories using Eq. (33) and (34) are used to determine the frequency response using Eq. (18). Hence the frequency response corresponding to both the laminated beam theories are determined.

4. NUMERICAL RESULTS AND DISCUSSION

The linear vibration (L) of Eq. (6) and non-linear vibration (NL) of Eq. (19) of the beam are studied using the geometrical and material properties given in Table 1. The laminate configuration considered is called as LC1 which is a laminate with ([0]_10)s configuration.
The objective of the present work is to compare the relative differences between the nonlinear motion characteristics of composite beams based on the two laminated beam theories. Frequency-response curves corresponding to a particular mode obtained here could not be validated quantitatively since such information is not available in the literature for composite beams. Therefore, a qualitative comparison has been made by comparing the results obtained using the two theories between themselves, and also considering for the comparison purpose the parts of the frequency-response curve that are sufficiently away from resonance condition so as to be comparable to linear response. In addition, the linearized part of the present solution is considered for comparison with linear analytical and finite element solutions. Hence, a partial validation of the present formulation is provided. The linear response of the beam is calculated by three different methods i.e. linear deflection of the beam given by Eq. (6), finite element analysis using ANSYS (BEAM188, 100 elements), and a special case of non-linear vibration response of the beam (by neglecting non-linear effect in Eq. (18)). Results are compared and plotted in Fig. 2. The frequency-response curves with variation of excitation force is illustrated in Fig. 3. The non-linear response is close to the linear response at $\sigma = \pm 20$ and that as the beam approaches resonance from this point, the beam exhibits significant non-linearity depending upon different excitation forces. Due to the hardening behaviour of

<table>
<thead>
<tr>
<th>Properties of the composite beam</th>
<th>Length (m)</th>
<th>Individual ply thickness (m)</th>
<th>Breadth (m)</th>
<th>Height of laminate (m)</th>
<th>Density (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>0.65</td>
<td>0.00075</td>
<td>0.015</td>
<td>0.015</td>
<td>1480</td>
</tr>
<tr>
<td>Individual ply thickness</td>
<td>0.65</td>
<td>0.00075</td>
<td>0.015</td>
<td>0.015</td>
<td>1480</td>
</tr>
<tr>
<td>Longitudinal modulus (E$^1$)</td>
<td>113.9 GPa</td>
<td>E$^1$=E$^2$</td>
<td>7.985 GPa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Transverse modulus (E$^2$)</td>
<td>7.985 GPa</td>
<td>E$^1$=E$^2$</td>
<td>7.985 GPa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>In-plane shear modulus (G$^{12}$)</td>
<td>3.138 GPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-plane shear modulus (G$^{23}$)</td>
<td>2.615 GPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Table 1: Properties of the composite beam |

Figure 2. Validation for linear part of response of the beam near first natural frequency

Figure 3. Variation of frequency-response curve with amplitude of excitation

Table 2: Response of the beam near resonance

<table>
<thead>
<tr>
<th>Detuning parameter ($\sigma$)</th>
<th>Force = 20 N</th>
<th>Force = 15 N</th>
<th>Force = 10 N</th>
</tr>
</thead>
<tbody>
<tr>
<td>NL, $\sigma = +20$</td>
<td>0.0662</td>
<td>0.0492</td>
<td>0.0326</td>
</tr>
<tr>
<td>L, $\sigma = 20$</td>
<td>0.0650</td>
<td>0.0487</td>
<td>0.0325</td>
</tr>
<tr>
<td>NL, $\sigma = -20$</td>
<td>0.0639</td>
<td>0.0483</td>
<td>0.0323</td>
</tr>
</tbody>
</table>

It can be inferred from the Fig. 3 that at the same excitation frequency, the beam has larger amplitude for larger excitation force. It can be observed from Table 2 that the non-linear response is close to the linear response at $\sigma = \pm 20$ and that as the beam approaches resonance from this point, the beam exhibits significant non-linearity depending upon different excitation forces. Due to the hardening behaviour of
the composite beam system, after passage through resonance, at \( \sigma \) values close to +10 the jump phenomenon occurs. This results in three possible amplitudes of motion at excitation frequencies corresponding to these \( \sigma \) values.

The dependence of non-linearity on the slenderness ratio \( S \) (length over radius of gyration) for \( f_z = 20N \) is shown in Fig. 4. Different values of slenderness ratio (i.e., \( S = 25, S = 50, S = 100, S = 150 \)) are considered for aspect ratio of 1 and its effect on beam non-linearity is evaluated. The dependence of beam non-linearity on the aspect ratio \( A \) (width over height) for \( f_z = 20N \) is shown in Fig. 5. Different values of aspect ratio (i.e., \( A = 2, A = 1.5, A = 1.2, A = 1 \)) are considered for slenderness ratio of 150 and its effect on beam non-linearity is evaluated.

![Figure 4. Effect of slenderness ratio on frequency-response curve of the first mode](image1.png)

![Figure 5. Effect of aspect ratio on the frequency-response curve of the first mode](image2.png)

Figure 4 shows that the beam non-linearity varies significantly from short beam (length/width < 15, i.e., \( S < 50 \)) to long beam (length/width > 15, i.e. \( S > 50 \)). It can be observed that the long beam behaves with less non-linearity near the fundamental frequency compared to the short beam and furthermore, the beam non-linearity increases as the slenderness ratio decreases. Short beam behaves like a hard spring and consequently shows more non-linearity whereas long beam behaves as soft spring and its behaviour tends more towards linear curve.

Figure 5 shows the dependence of beam non-linearity on the aspect ratio. As aspect ratio decreases the beam non-linearity increases. Thin beam behaves more linearly near the resonance compared to thick beam. Thick beam behaves like a hard spring and consequently shows more non-linearity whereas thin beam behaves as soft spring and its behaviour tends more towards linear curve.

Dependence of non-linearity on the laminated composite beams are evaluated using cylindrical bending theory (CB) and 1D beam theory (BT) in the Fig. 6. The natural frequencies obtained based on both the theories using Eq. (35) and (36) are used to determine the frequency response using Eq. (18). The laminate configurations considered are: 1) LC1 which is the laminate with \(([0]_{10})s\) configuration, 2) LC2 \(([0/\pm 45]_{2s})\), 3) LC3 \(([\pm 30/90/\pm 45]_{2s})\), 4) LC4 \(([90]_{10})s\).

![Figure 6. Effect of non-linearity on frequency-response curves of different laminated composite beams](image3.png)
As seen in Fig. 6 the laminate configuration LC1 behaves most non-linearly compared to all other laminate configurations and LC4 behaves most linearly. Both theories, the cylindrical bending theory and 1D beam theory predict the same trends. However, for the case of LC1 and LC2 the difference between both theories is very small because of cross-ply laminate configuration. Whereas, for the case of laminate configurations LC3 and LC4, the cylindrical bending theory shows more non-linearity compared to the 1D beam theory because of angle-ply laminate configuration.

The major difference between the cylindrical bending theory and 1D beam theory for symmetrical laminates is in the bending stiffness term. The term $D_{11}$ in cylindrical bending theory and $D_{11}^*$ in 1D beam theory shows numerically the same results in the case of cross-ply laminates when $v_{xy}^b$ term is small in Eq. (32) whereas the largest difference occurs in the case of angle-ply laminate for which the value of $v_{xy}^b$ is large.

The laminate configuration LC1 is the strongest in terms of flexural stiffness coefficient $D_{11}$ and bending stiffness term ($1/D_{11}$). The laminate configuration LC2 is the second largest, LC3 and LC4 are third and fourth largest respectively.

5. CONCLUSION
The present study examined the effect of geometrical non-linearity on the vibrations of simply-supported laminated composite beams subjected to harmonic loading. The following conclusions can be drawn:

1. The comparison of excitation frequencies at which the composite beam starts exhibiting significant nonlinearity for different slenderness ratio and aspect ratio values shows that the short and thick beam behaves like a hard spring and consequently shows more non-linearity whereas the long and thin beam behaves as soft spring and shows less non-linearity.
2. Cylindrical bending theory predicts the non-linearity in composite laminates more precisely because it considers the coupling between stretching and bending of the laminate whereas the one-dimensional laminated beam theory considers only the bending of symmetric laminates through the corresponding stiffness matrix.

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