This paper examines the a-priori assignment of polynomial order in the $p$-version of the FEM for the efficient simulation of time-harmonic acoustic problems. An error indicator is proposed, which accounts not only for the local interpolation error, but also for the accumulation of phase errors (i.e. the so-called pollution effect). The finite element mesh is traversed using a variant of the breadth first search, a strategy developed for graph numbering. The local phase error is computed from a dispersion analysis performed on a one-dimensional element of equivalent size. The proposed method is verified on several academic Helmholtz problems including non-uniform meshes.

Keywords: Finite element, Helmholtz, dispersion, pollution error

With $hp$-FEM, it is possible to adjust the size of the finite element as well as their interpolation order. Ideally, allocating the element order should be done before the calculation is performed. This requires the use of a priori error indicator. For instance in the technique from [1] for the Helmholtz equation, the polynomial order is selected based on the local sound speed, frequency and element size. This approach is purely local and one difficulty is to account for the accumulation of dispersion error as the waves propagate through the computational domain. This is called the pollution effect and is a global effect that depends on the size and shape of the computational domain [2].

In this paper we propose a simple approach to estimate the pollution effect by using tools from graph theory as well as the dispersion properties of individual elements. We begin by showing that in one-dimension the pollution can be precisely predicted by accumulating dispersion error from one element to the next. We then show how this approach can be extended to several dimensions and illustrate the procedure on a simple test case.
1. One-dimensional analysis

1.1 Error indicator

We begin by examining the relation between a dispersion error and the global $L^2$ error measured over a given propagation distance $L$ in one dimension. Dispersion analysis provides a measure of the phase error over one wavelength for an infinite periodic mesh with element size $h$. We consider a right-propagating one-dimensional wave of amplitude $A$ and theoretical wavenumber $k$: $f(x) = A \exp(-ikx)$. In the absence of interpolation error, the numerical solution takes the form:

$$\tilde{f}(x) = A \exp \left( -i \int_0^x \tilde{k}(\xi)d\xi \right),$$

where $\tilde{k}(x)$ is the numerical wavenumber. The resulting numerical error is given by

$$f(x) - \tilde{f}(x) = A \exp(-ikx) \left\{1 - \exp \left( -i \int_0^x (\tilde{k}(\xi) - k) \, d\xi \right)\right\}.$$  \hspace{1cm} (2)

The corresponding $L^2$ numerical error, accumulated over the propagation region $x \in [0; L]$ can be written:

$$\int_0^L \left| f(x) - \tilde{f}(x) \right|^2 \, dx = A^2 \int_0^L \left| 1 - \exp(-i\phi(x)) \right|^2 \, dx, \quad \text{with } \phi(x) = \int_0^x \delta_k(\xi)d\xi,$$  \hspace{1cm} (3)

where $\delta_k = \tilde{k} - k$. $\phi(x)$ denotes the accumulated phase error. We consider a piecewise constant variation of $\delta_k$ over the computational domain. As a result, the accumulated phase error $\phi(x)$ may be considered continuous, piecewise linear. The domain $[0; L]$ is now split into finite elements $\Omega_e : [x_e; x_{e+1}]$, and the $L^2$ error is obtained by summing the element contributions as

$$\int_0^L \left| f(x) - \tilde{f}(x) \right|^2 \, dx = A^2 \sum_{e} \int_{x_e}^{x_{e+1}} \left| 1 - \exp(-i\phi(x)) \right|^2 \, dx.$$  \hspace{1cm} (4)

In element $e$, $\phi(x)$ will be of the form

$$\phi(x) = \psi_{e-1} + \int_{x_e}^{x} \delta_k d\xi = \psi_e + \delta_k(x - x_e), \quad \text{with } \psi_e = \sum_{e'=1}^{e} \delta_{k_e} h_{e'},$$

with $h_e$ the length of element $e$. The relative $L^2$ error is defined as

$$\varepsilon_{L^2}^2 = \int_0^L \left| f(x) - \tilde{f}(x) \right|^2 \, dx / \int_0^L |f(x)|^2 \, dx,$$

which may again be obtained by summing the element contributions:

$$\varepsilon_{L^2}^2 = \sum_{e} \varepsilon_{e}^2, \quad \text{with } \varepsilon_{e}^2 = \frac{1}{L} \int_{x_e}^{x_{e+1}} \left| 1 - \exp(-i\phi(x)) \right|^2 \, dx = \frac{2h_e}{L} \left\{1 - \frac{\sin(\psi_e) - \sin(\psi_{e-1})}{\delta_{k_e} h_{e}}\right\}.$$  \hspace{1cm} (5)

This simple error indicator is now tested for the propagation of a one-dimensional wave over a computational domain of length $L$, with varying element size distribution and interpolation orders $p$.  

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1.2 Test case

We now consider a simple one-dimensional test case to assess the global numerical error observed in practice, and to examine the robustness of the proposed simple error indicator. To create a single propagating wave \( f_{\text{exact}} = e^{-i k x} \) in the computational domain, Robin-type boundary conditions are used at both ends of the computational domain. The one-dimensional grid is obtained using Gmsh [3], with different mesh sizes prescribed at \( x = 0 \) and \( x = L \), denoted respectively \( h_0 \) and \( h_L \). We define the minimum resolution (i.e. the number of degrees of freedom per wavelength) as (see [4])

\[
d_{\lambda_{\text{min}}} = \frac{2\pi}{k_{\text{max}}(h_0, h_L)}
p.
\]

![Figure 1](image1.png)

**Figure 1:** Numerical (blue) and analytical (black) solution for the 1D problem when applying the mesh refinement at \( x = 0 \) (a) or at \( x = L \), for a domain size \( L = 20 \). The actual \( L^2 \) error measured for the two problems are 13.1\% ad 21.9\% respectively. The location of the nodes is indicated with crosses.

As a first application, Figure 1(a) presents the numerical solution obtained using linear elements \( (p = 1) \), on a grid with a strong refinement applied at \( x = 0 \) \( (h_0 = 0.1 \) and \( h_L = 1) \) for a domain length of \( L = 20 \). This corresponds to a minimum resolution rate of \( d_{\lambda_{\text{min}}} = 2\pi \). The numerical solution behaves well in the left part of the computational domain but starts to degrade as the mesh size is relaxed. A phase shift is noticeable when reaching the end of the domain \( (x = L) \), which indicates that the dispersion error is likely to dominate the total error. The same numerical experiment is conducted, but this time by applying the mesh refinement at \( x = L \) \( (h_0 = 0.1, h_L = 1) \). It is striking to note that the two grids yield very different numerical solutions. The second scenario is visually less favorable, because it allows the large dispersion errors induced in the larger elements to be propagated over a larger distance. This is confirmed by the error analysis, with an actual error of \( E_2 = 13.1\% \) and \( E_2 = 25.3\% \) measured...
respectively for the $x = 0$ and $x = L$ refinements. The proposed error indicator approximates well the resulting error with a global predicted error of $\varepsilon_{L^2} = 15.0\%$ and $\varepsilon_{L^2} = 25.3\%$ respectively for the two grids, with a marginal contribution from the local interpolation error (which is equal on the two grids) with $\varepsilon_{int}^{L} = 3.74\%$.

It is worth emphasizing that the process with which the phase error accumulates is direction dependent. The proposed error indicator will require the mesh to be traversed, but the actual result (for the dispersion error contribution) will depend on the direction which is chosen.

We now examine the results obtained for a fixed element size distribution, with $h_0 = 1$ and $h_L = 0.1$ and a varying domain length.

The frequency is set to $\omega = p$ which ensures a constant minimum resolution rate of $d_{\lambda_{\text{min}}} = 2\pi$ for each interpolation order $p$. The domain size is progressively varied from $L = 1$ to $L = 512$. The predicted and actual numerical errors obtained for $p = 1$ to $p = 4$ are shown in Figure 2. The proposed error indicator for the accumulation of dispersion error is able to predict accurately the actual $L^2$ error. It is clear that the dispersion error dominates in the cases considered here, as expected with a resolution $d_{\lambda_{\text{min}}} = 2\pi$.

![Figure 2](image-url)  
Figure 2: Actual and predicted error for a 1D problem with non-uniform mesh, with a factor 10 refinement applied at $x = L$ ($h_0 = 1$, $h_L = 0.1$) and $\omega = p$ for a varying domain length $L$. 
2. Two-dimensional analysis

To extend the one-dimensional approach outlined above to several dimensions, we propose here to explore the mesh topology using a strategy developed for graph numbering. To illustrate the proposed approach, a simple two-dimensional in-duct propagation problem is considered. It consists in a straight duct of length $L = 6$ and height $H = 1$, on which a secondary circular tube is attached, see Figure 3c. This type of device is often used in acoustics, and referred to as a Herschel-Quincke tube. This test case involves constructive and destructive interference and multiple wave propagation paths. At the main duct inlet (on the left), a plane wave mode is injected using a Robin boundary condition. At the outlet (on the right), a cartesian PML [5] is employed to absorb any outgoing wave. On all the other boundaries, hard-wall conditions are applied. To examine the accumulation of phase errors, a serial configuration of $N$ tubes is considered. Figure 3c presents a case with $N = 4$ tubes. For this example, a non-uniform mesh size is chosen, with $h = 0.2$ in the main duct and $h = 0.3$ in the resonators. Standard 6-node, triangular, quadratic elements are used ($p = 2$).

First note that the accumulated dispersion error $\psi_e$ is defined in each element (other definitions could be used but will not be considered in this short paper). As a consequence we consider the so-called natural associate graph (NAG) of the finite element model [6]. This is the graph whose nodes are in one to one correspondence with the triangular finite elements and two nodes are connected if the corresponding elements have a common boundary (i.e. a common edge).

Note also in equation (5) that the dispersion error $\psi_e$ accumulated in one element is the sum of the dispersion error $\delta_k h_e$ in this element and the dispersion error accumulated in all the previous elements considered in the mesh. The aim of the proposed method is to follow a heuristic that mimics the same process of accumulation by traversing the graph.

The NAG is traversed using the breadth first search (BFS) algorithm. The BFS starts at the tree root (here we arbitrarily consider the lower left triangle element), and explores all of the neighbor vertices at the present depth prior to moving on to the vertices at the next depth level. The complexity of this algorithm remains low since every vertex and every edge will be explored in the worst case. This provides a numbering of the elements as well as a definition of successive fronts that propagates from a starting elements throughout the mesh. The accumulation of dispersion error is then calculated by following the fronts defined by the BFS. For an element $e$ which is part of the $n$th front, we calculate the local dispersion error $\delta_k h_e$ incurred within this element. We then consider the connected elements that are part of the previous front, i.e. the elements where we have already calculated $\psi_e$. Among these predecessors we pick the element $e'$ which has the largest accumulated error $\psi_{e'}$. We calculate the accumulated error in element $e$ as $\psi_e = \psi_{e'} + \delta_k h_e$. We repeat this process for every element in front $n$, and then for every front, so as to obtain $\psi_e$ calculated in every element of the mesh.

Figure 3a presents the estimation of the local phase errors $\delta_k h_e$ in each element in logarithmic scale obtained at $kH = 4$. The latter is obtained from a numerical dispersion analysis performed on a line element of equivalent size $h_e$, where $h_e$ designates the average edge length for each triangle [4]. Larger phase errors are typically obtained for larger elements, located in the resonators.

Figure 3b presents the estimation of the phase error accumulation $\psi_e$, obtained by traversing the NAG graph using the BFS, starting from the bottom left corner. This provides an estimates of dispersion error that can be accumulated over the computational domain. From this we can identify the element where the accumulated error is the largest.

We can also identify the critical path that led to this largest error. Starting from the element $e$ with the largest $\psi_e$ we can find its predecessor $e'$ that contributed to the dispersion error in element $e$. We can then identify the predecessor of the predecessor, and so on, until the complete path between the first element and the worst element is identified. This critical path is shown in 3b. It consists of 236 consecutive
elements. Note that the last element in the critical path (i.e. corresponding to the element with largest phase error) is not a leaf vertex, because the maximum graph depth is found to be slightly higher, with $d_{\text{max}} = 238$. The critical path goes through each of the upper branches of the duct where the elements are larger (therefore where the dispersion error is more important).

Figure [4] show the evolution of the local dispersion error $\delta_k$, $h_e$ and the accumulated error $\psi_e$ along the critical path. The local errors are periodically increasing when the critical path traverses the larger elements located in the four resonators.

For the solution shown in Figure [3c], the error indicator predicts $\varepsilon_{L^2}^{\text{pol}} = 11.6\%$ and $\varepsilon_{L^2}^{\text{int}} = 1.9\%$, respectively for the pollution and the interpolation error, which is typical of the pre-asymptotic regime, where the dispersion error dominates. The actual $L^2$ error is found to be of the same order of magnitude, with $E_2 = 12.42\%$.

Finally, we can examine more closely the behaviour of the error indicator on a series of meshes at several frequencies. The frequency is increased from $\omega = 2.5$ to $\omega = 20$. The mesh size, chosen uniform out of simplicity, is decreased accordingly, so as to maintain a constant engineering resolution of 12.5 DOFs per wavelength for all test cases. To examine the influence of the domain size, the number of tubes $N$ is gradually increased from $N = 1$ to $N = 16$. The total length of the main duct ranges from 6 to 94. The actual numerical error and the predicted error are summarized in table 1. Overall the evolution of the numerical error as the different parameters are varied is followed quite well by the proposed error indicator.

<table>
<thead>
<tr>
<th>$\omega$, $h$</th>
<th>N=1</th>
<th>N=2</th>
<th>N=4</th>
<th>N=8</th>
<th>N=16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega = 2.5$, $h = 0.4$</td>
<td>1.31% (2.25%)</td>
<td>2.3% (3.1%)</td>
<td>5.1% (4.0%)</td>
<td>8.8% (8.3%)</td>
<td>16.3% (16.0%)</td>
</tr>
<tr>
<td>$\omega = 5$, $h = 0.2$</td>
<td>2.7% (4.9%)</td>
<td>4.6% (7.3%)</td>
<td>11.5% (13.6%)</td>
<td>18.9% (20.8%)</td>
<td>52.4% (41%)</td>
</tr>
<tr>
<td>$\omega = 10$, $h = 0.1$</td>
<td>3.9% (8.9%)</td>
<td>5.1% (14%)</td>
<td>9.3% (22%)</td>
<td>31.2% (41.2%)</td>
<td>-</td>
</tr>
<tr>
<td>$\omega = 20$, $h = 0.05$</td>
<td>6.2% (15.7%)</td>
<td>13.5% (24%)</td>
<td>36% (48%)</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Actual and estimated (in brackets) $L^2$ error for increasing number of serial ducts $N$ using quadratic elements $p = 2$ with a fixed resolution rate of 12.5 degrees of freedom per wavelength.
(a) Local phase error, $\delta_{k_h}e$

(b) Accumulated phase error, $\psi_e = \sum_{e'=1}^{e} \delta_{k_h} h_{e'}$ using BFS and critical path

(c) Numerical solution for incident mode $m = 0$

Figure 3: Results of the a-priori error indicator on the Herschel Quincke problem (with 4 ducts) with mesh size $h = 0.2$ using quadratic element $p = 2$.

Figure 4: Local and accumulated phase errors obtained along the critical path on the Herschel Quincke problem (with 4 ducts) at $kH = 4$ with mesh size $h = 0.2$ using quadratic elements (solid lines) and cubic (dashed lines) elements.
REFERENCES


