A numerical optimization method is performed on a fictive but realistic porous material with a continuous through the thickness gradient of porosity. The porous material is made of ordered unit cell with parametric geometrical dimensions. The Johnson-Champoux-Allard-Lafarge parameters of any geometrical unit cell is computed by FEM method. These parameters are used as input for the prediction of the acoustic properties of the graded material. A conjugate gradient algorithm eventually creates the best micro-geometrical continuous gradient with the optimal acoustic absorption. The acoustic performances of the continuous graded material are discussed with respect to the optimized non-graded material. All the investigations are made on the same unit cells. The numerical results show a significant improvement of the targeted acoustic absorption when introducing the gradient of properties.

Keywords: acoustic, porous, gradient, optimization

1. Introduction

Over the past decade, open-cell acoustic materials have been widely investigated for broadband noise absorption applications. But the unwanted low-frequency noise remains a major challenge. Moreover some of these materials present large absorption variations within the frequency space. Additionally, perfect absorption is only possible at a single frequency and the absorption curve presents ripples in frequency. Introducing a through-the-thickness gradient of properties is one of the most promising avenue to widen the absorption range, lower perfect absorption frequency or control the surface impedance. A trough the thickness gradient may not only enable absorption at lower frequencies but also soften the ripples observed for usual homogenous porous materials. It has been demonstrated that multilayered materials can have a better absorption than homogeneous materials, in a given frequency range. However, no continuous gradient optimization method has been detailed in the literature.
The main purpose of this work is to propose a numerical optimization method of a continuous gradient of properties under normal incidence configuration and with a rigid backing termination. The method is applied to a periodic fictive material as an example.

2. Description of the problem

A schematic representation of the problem is depicted on Fig.1. The porous slab is considered infinitely long in the y and z directions and has a thickness L in the x direction. Both faces are flat and parallel. The faces are designated \( \Gamma_0 \) and \( \Gamma_L \) with x coordinates 0 and L, respectively. The sample is rigidly backed; then impedance of \( \Gamma_L \) is infinite. The slab’s porosity inhomogeneity occurs in the x direction. The incident plane wave is invariant along z and propagates in the positive x direction. In this way, the problem depends only on the x coordinate. The surrounding and saturating fluid is air and has the characteristic impedance \( Z_0 \). The \( e^{j\omega t} \) convention is used which leads to the normal incident plane wave expression \( p^i = e^{j(\omega t - k_0 x)} \) wherein \( k_0 \) is the incident wavenumber. The porous slab domain is denoted \([p]\), the air facing \( \Gamma_0 \) is denoted \([a]\).

![Figure 1: Schematic representation of the problem. \( p^i \) and \( p^r \) are the incident and reflected wave respectively.](image)

3. Porous medium: micro-lattice

3.1 Geometry description

The numerically optimized porous material is a fictive but realistic micro-lattice presented on Fig.2. Its constitutive unit cell is the junction of two square section orthogonal rods aligned with y and z directions. The unit cell can be described by two parameters: the rods width \( C \) and their spacing. Other equivalent parameters are the porosity \( \phi \) and the adimensional rods \( Step \). The latter is the spacing normalized by the rods width. The porosity of a given rods plane is expressed as

\[
\phi = 1 - \frac{1}{\text{Step}}.
\]  

(1)

In this study, the micro-structure gradient is obtained by varying the \( \text{Step} \) which in turn leads to the variation of \( \phi \). The rods width \( C \) is set to 250 \( \mu \text{m} \) and the the porosity \( \phi \) is bounded between \( \phi_{\min} = 0.33 \) (\( \text{Step} = 1.5 \)) and \( \phi_{\max} = 0.96 \) (\( \text{Step} = 25 \)).
The material is not isotropic but as the incidence wave is normal to the slabs and as the material main axis are aligned with the slab’s one, only the out-of-plane properties of the material impact the waves propagation. In the following, the slab will be considered isotropic.

Figure 2: Porous material micro-structure. The blue box delimits an unit cell. The acoustic wave propagates along x.

3.2 Equivalent fluid model / Two-Scale Asymptotic Method

The porous medium can be assumed to behave as an equivalent fluid [1]. The viscous and thermal losses in the pores are accounted in the equivalent density $\rho_{eq}$ and in the equivalent Bulk modulus $K_{eq}$. These quantities are complex, frequency and $x$ dependent because of the gradient of porosity. They are defined according to the semi-phenomenological Johnson-Champoux-Allard-Lafarge (JCAL) model [9, 10, 4]. According to this model, the equivalent density can be expressed in terms of air parameters and six parameters depending on the porous medium, namely the porosity $\phi$, the geometrical tortuosity $\alpha_{\infty}$, the viscous characteristic length $\Lambda$, the thermal characteristic length $\Lambda'$, the visco-static permeability $q_0$ and the thermo-static permeability $q'_0$.

The JCAL parameters are computed for a given rods width $C$ and a discrete number of porosities including $\phi_{\text{min}}$ and $\phi_{\text{max}}$. They are numerically obtained using a Two-Scale Asymptotic Method [2] implemented in a FEM code [11]. This method solves fundamental equations in the fluid domain embedding the rigid skeleton and retrieves the JCAL parameters after a few fluid and fluid-solid domains integrations. The equations are solved in the domain limited by representative unit cell (Fig.2) witch is 3D periodic. Then, each JCAL porosity dependent parameter is obtained by interpolating a continuous function from the discrete set of computed values. The FEM mesh must be fine enough to smooth the porosity dependence of the other JCAL parameters. This condition is a prerequisite for the application of the gradient optimization method.

4. Acoustic waves propagation in graded porous material

Using the alternative Biot’s formulation [3], De Ryck et al. derived the equations of motion in a macroscopically inhomogeneous porous material under the rigid frame approximation [5]. In normal
incidence configuration, the equations can be expressed as follows
\[ \rho_{eq} \frac{\partial V[^p]}{\partial t} = -\frac{\partial p[^p]}{\partial x} \]  
(2)
\[ -\frac{1}{K_{eq}} \frac{\partial p[^p]}{\partial t} = \frac{\partial V[^p]}{\partial x}, \]  
(3)
where \( p \) is the fluid pressure in the material pores and \( V \) the equivalent normal component of the fluid velocity in the interconnected pores.

By means of a state vector formalism and wave splitting-transfer Green functions \( G^\pm \), the following differential equations system can be set:
\[ \frac{\partial}{\partial x} \begin{pmatrix} G^+(x,\omega) \\ G^-(x,\omega) \end{pmatrix} = \begin{pmatrix} A^+ & A^- \\ -A^- & -A^+ \end{pmatrix} \begin{pmatrix} G^+(x,\omega) \\ G^-(x,\omega) \end{pmatrix}, \]  
(4)
where
\[ A^\pm = \frac{j\omega}{2} \left( \frac{Z_0}{K_{eq}} \pm \rho_{eq} \right). \]  
(5)
The solution of Eq. (4) is found by integrating from \( x = L \) where the boundary conditions are known for \( x = 0 \). The boundary condition is \( R(L,\omega) = 1 \) leads for the Green’s function to:
\[ G^+(L,\omega) = G^-(L,\omega) \]  
(6)
After solving Eq. (4) together with BC given by Eq. (6), the reflection coefficient is
\[ R(\omega) = \frac{G^-(0,\omega)}{G^+(0,\omega)}, \]  
(7)
Moreover, the following equation can be set using the same formalism:
\[ \frac{\partial}{\partial x} R = 2A^+ R + A^- (1 + R^2). \]  
(8)

5. Gradient optimization

The described method is widely inspired from the work of De Ryck et al. \[6\]. It tackles the JCAL graded profile reconstruction problem by means of a conjugate gradient algorithm. The present work adapted the proposed methodology to absorption optimization.

5.1 Conjugate gradient algorithm

The objective of the gradient optimization is to maximize the absorption coefficient \( \alpha = 1 - |R|^2 \) of the porous slab within a frequency region, by tuning the gradient of porosity. This objective is mathematically defined by means of a cost function \( J \) that has to be minimized:
\[ J (\phi(x)) = \sum_\omega W(\omega) |R(\omega)|^2, \]  
(9)
where \( W(\omega) \) the frequency weighting, \( \phi \) varies along the slab’s thickness \( x = [0; L] \) and is the subject of the optimization.
The Conjugate Gradient method is a well known iterative algorithm \cite{7}. The steps are the following:

**Step 0 - Initial guess:** $\phi^{(0)}$ is given a $x$ dependent value.

**Step 1 - First search direction:** Set $i = 0$. Compute $R(x, \omega) \forall x \in [0; L]$ by means of Eq.\cite{7} and the gradient of the cost function (more details below):

$$G(\phi^{(0)}) = \frac{dJ}{d\phi^{(0)}}.$$  \hspace{1cm} (10)

Set $D^{(0)} = G(\phi^{(0)})$ wherein $D^{(i)}$ is the search direction of iteration $i$.

**Step 2 - Optimal step size:** Compute $\lambda_i$ minimizing the cost function

$$J(\phi^{(i)} - \lambda_i D^{(i)}) = \min_{\lambda \in \mathbb{R}^n} (J(\phi^{(i)} - \lambda D^{(i)})).$$  \hspace{1cm} (11)

**Step 3 - Update $\phi(x)$:**

$$\phi^{(i+1)}(x) = \phi^{(i)}(x) - \lambda_i D^{(i)}(x).$$  \hspace{1cm} (12)

**Step 4 - Compute the new search direction:**

$$G^{(i+1)} = G(\phi^{(i+1)}),$$  \hspace{1cm} (13)

$$D^{(i+1)} = G^{(i+1)} + \gamma_i D^{(i)}$$  \hspace{1cm} (14)

where $\gamma_i$ is computed by the Polak-Ribiere formula:

$$\gamma_i = \max \left( \frac{G^{(i+1)}T G^{(i+1)} - G^{(i+1)}T G^{(i)}}{G^{(i)}T G^{(i)}}, 0 \right).$$  \hspace{1cm} (15)

If $i$ is higher than the maximum number of iterations then the algorithm stops. Otherwise $i = i + 1$ and loop to Step 2.

### 5.2 Cost function gradient

This section details how to compute Eq.\cite{10}.

Let $\delta \phi$ be a small variation of $\phi$. It induces a infinitesimal variation of $R(x, \omega, \phi)$ equal to $\delta R(x, \omega, \phi) = R(x, \omega, \phi + \delta \phi) - R(x, \omega, \phi)$. The surface impedance on $\Gamma_L$ is infinite; this boundary condition does not depend of the slab’s porosity. Then,

$$\delta R(x = L, \omega, \phi) = 0.$$  \hspace{1cm} (16)

Perturbing Eq.\cite{8} leads to

\[ \frac{\partial}{\partial x} \delta R - 2(A^+ + A^- R) \delta R = 2R \delta A^+ + (1 + R^2) \delta A^- . \]  \hspace{1cm} (17)

wherein

$$\delta A^\pm = \frac{j \omega}{2} (Z_0 \delta K_{\text{eq}}^{-1} \pm Z_0^{-1} \delta \rho_{\text{eq}}) .$$  \hspace{1cm} (18)

The derivative of the equivalent fluid density and bulk modulus are

$$\delta \rho_{\text{eq}} = \frac{\partial \rho_{\text{eq}}}{\partial \phi} \delta \phi .$$  \hspace{1cm} (19)

$$\delta K_{\text{eq}}^{-1} = \frac{\partial K_{\text{eq}}^{-1}}{\partial \phi} \delta \phi .$$  \hspace{1cm} (20)
The equivalent fluid density and bulk modulus are expressed in terms of JCAL parameters which depend in a non-analytic way on the porosity. Then, there is no analytic expression of Eqs. (19,20). They have to be computed by means of the derivation definition:

$$\frac{\partial f}{\partial \phi} = \lim_{\delta \phi \to 0} \frac{f(\phi + \delta \phi) - f(\phi)}{\delta \phi}. \quad (21)$$

The variation of the cost function perturbated by $\delta \phi$ takes the form [13]

$$\delta J(\phi) = 2 \text{Re} \sum_\omega u_R(0, \omega) \delta R(0, \omega), \quad (22)$$

wherein, setting * as the complex conjugate notation,

$$u_R(0, \omega) = W(\omega) R(\omega)^* \quad (23)$$

The following integration is obtained considering the boundary condition Eq. (16):

$$\int_0^L \frac{\partial}{\partial x} (u_R(x, \omega) \delta R(x, \omega)) = -u_R(0, \omega) \delta R(0, \omega) \quad (24)$$

The right term of this equation is included in Eq. (22). The left term’s integrand can be written from Eq. (8) as

$$\frac{\partial}{\partial x} u_R \delta R = \delta R \left( \frac{\partial u_R}{\partial x} + 2u_R(A^+ + A^-) \right) + u_R (2R \delta A^+ + (1 + R^2) \delta A^-), \quad (25)$$

where $u_R(x, \omega)$ is an arbitrary function chosen in such a way that the $\delta R$ dependency is eliminated. In order to do this, it must satisfy

$$\frac{\partial u_R}{\partial x} = -2(A^+ + A^-) u_R \quad (26)$$

Equation (25) reduces to

$$\frac{\partial}{\partial x} u_R \delta R = u_R (2R \delta A^+ + (1 + R^2) \delta A^-) \quad (27)$$

A new expression of the variation of the cost function is then obtained by combining Eqs. (22,24,27):

$$\delta J(\phi) = -2 \text{Re} \sum_\omega \int_0^L u_R(2R \delta A^+ + (1 + R^2) \delta A^-). \quad (28)$$

This variation can also be simply expressed as

$$\delta J(\phi) = \int_0^L \frac{\partial J}{\partial \phi} \delta \phi. \quad (29)$$

The identification of Eq.(28) with Eq.(29) and replacing the derivatives by their expression, leads to

$$\frac{\partial J}{\partial \phi}(x) = - \text{Re} \sum_\omega j \omega u_R(x, \omega) \left( Z_0(1 + R(x, \omega))^2 \frac{\partial K_{\text{eq}}^{-1}}{\partial \phi_i} - \frac{(1 - R(x, \omega))^2 \partial \rho_{\text{eq}}}{Z_0} \frac{\partial \phi_i}{\partial \phi} \right). \quad (30)$$
6. Optimization example

An optimization is presented in Fig. 3. The slab’s thickness is equal to 30 mm. As stated above, the rods width is set equal to 250 $\mu$m and the porosity can vary between 0.33 and 0.96. An homogeneous micro-lattice, perfectly absorbing at the so-called "quarter wave-length resonance" frequency is given as a comparison reference. The frequency weighting function $W$ is a pass band filter equal to one between 2000 and 5000 Hz and zero elsewhere.

The gradient of porosity greatly increases the absorption coefficient in the frequency band of interest. The first maximum of absorption is shifted towards the high frequencies while the second one is is shifted towards the low frequencies. Both of them are placed in the frequency region delimited by $W$ and are widened.

The continuous graded porosity profile presents several oscillations, alternating high and low values. The minimum authorized porosity is reached around $x = 20$ mm.

![Figure 3: 30 mm thick micro-lattice having 250 $\mu$m rods width. (a) Absorption coefficients of the graded material optimized between 2000 and 5000 Hz (dashed line) and of the homogeneous material (solid line). (b) Graded corresponding porosity profiles.](image)

7. Conclusion

A porosity gradient optimization algorithm has been adapted from an inverse characterization method. Its aim is to maximize an absorption coefficient within a frequency region. Preliminary results show an important increase of the absorption in the targeted zone. The optimized profile is not monotonic whereas most graded porous materials are. This tends to demonstrate the importance of the presented algorithm to get the best of the introduction of a through-the-thickness gradient. Future works will give better understanding of the optimized profiles and compare the performances of the algorithm with respect to a monotonic gradient optimization. They will also explore the low frequencies optimization.
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