Beginning with the familiar expression for frequency modulation (FM) synthesis, a system is examined/analyzed whereby the carrier frequency is modulated by “looping back” the output signal, with a feedback coefficient determining the amount of frequency deviation. “Loopback FM” is reminiscent of the previously coined “Feedback FM” technique, but with the feedback signal being applied to the carrier frequency (in the former) rather than the initial phase (as in the latter). Though both produce harmonic distortion, loopback FM has a less obvious phase modulation (PM) representation (the numerically advantageous implementation of FM) requiring integration of the instantaneous frequency, and also results in a shift of the sounding frequency which can be a potential source of frustration for some computer musicians. If the feedback coefficient is made time varying, the pitch will also change over time (glide) and further complications arise in the integration of frequency and PM representation. In this work the change of pitch is explored via an alternate representation of the system also having two variables, expressions for which are given in terms of corresponding FM/PM parameters. In addition to providing a mapping between desired pitch trajectories and FM parameters, this alternate representation allows for analytic (rather than numerical) integration of frequency, yielding a more precise expression for instantaneous phase in the PM representation. Also potentially musically useful is the ability to reverse the process by obtaining loopback FM parameters for specified pitch glide trajectories over time.

Keywords: frequency modulation, coupled modes, nonlinear coupling, pitch glide
can be made to oscillate with frequency $f$ Hz by first multiplying its real and imaginary parts with a power preserving unitary rotational matrix:

$$
\mathbf{r} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{bmatrix}
\begin{bmatrix}
\Re\{z(0)\} \\
\Im\{z(0)\}
\end{bmatrix},
$$

(2)

where the angle of rotation

$$
\theta = \omega T = 2\pi f / f_s,
$$

(3)
is the normalized frequency (radians/sample) at sampling rate $f_s = 1/T$. The components of the vector resulting from (2),

$$
\mathbf{r} = \begin{bmatrix}
A \cos \theta \cos \phi - A \sin \theta \sin \phi \\
A \sin \theta \cos \phi + A \cos \theta \sin \phi
\end{bmatrix} = \begin{bmatrix}
A \cos(\theta + \phi) \\
A \sin(\theta + \phi)
\end{bmatrix},
$$

(4)

serve as the real and imaginary parts of the new rotated point in the complex plane

$$
z(1) = \mathbf{r}(1) + j\mathbf{r}(2) = A (\cos(\theta + \phi) + j \sin(\theta + \phi)) = Ae^{j(\theta + \phi)} = e^{j\theta} Ae^{j\phi},
$$

(5)

showing that a single rotation by $\theta$ using a power-preserving unitary matrix may be equivalently accomplished with a unit-magnitude complex multiply. A regular rotation by $\theta$ every time sample $n = 0, 1, 2, \ldots, N - 1$ produces an oscillator that may be expressed by the complex exponential sinusoid

$$
z(n) = A(e^{j\theta})^n e^{j\phi} = Ae^{j(\omega n T + \phi)},
$$

(6)

having instantaneous phase $\theta_i(n) = \omega n T + \phi$. Substituting integer multiples of the sampling period $nT$ by time $t$ allows for continuous-time expressions showing the relationship between instantaneous frequency $\omega_i(t)$ and instantaneous phase $\theta_i(t)$:

$$
\omega_i(t) = \frac{d}{dt} \theta_i(t) \quad \text{and} \quad \theta_i(t) = \int_0^t \omega_i(t) \, dt,
$$

(7)

which, for static rotation angle $\theta$ given in (3), yields instantaneous frequency

$$
\omega_i(t) = \frac{d}{dt} (\omega t + \phi) = \omega
$$

(8)

and instantaneous phase

$$
\theta_i(t) = \int_0^t \omega \, dt = \omega t + C,
$$

(9)

where constant of integration $C = \phi$ may be viewed as the initial phase. In another example however, where the oscillator frequency is made to change linearly from $\omega_1$ to $\omega_2$ over $T_d$ seconds, the instantaneous frequency would be

$$
\omega_{i,\text{lin}}(t) = mt + \omega_1, \quad \text{where} \quad m = (\omega_2 - \omega_1) / T_d \quad \text{is the frequency slope},
$$

(10)

and the instantaneous phase is

$$
\theta_{i,\text{lin}}(t) = \int_0^t (mt + \omega_1) \, dt = \frac{m}{2} t^2 + \omega_1 t + C = \left(\frac{m}{2} t + \omega_1\right) t + \phi,
$$

(11)

perhaps notably, not $\omega_i(t) t + \phi$ as in the static case.
3. Frequency and Phase Modulation (FM/PM)

It is well known that the frequency and/or phase of an oscillator may be made to change sinusoidally, the basis of a synthesis technique pioneered by Chowning [1]. In frequency modulation (FM), a sinusoidal oscillator’s carrier frequency $\omega_c$ is modulated by a sinusoid having frequency $\omega_m$ and amplitude $d$, where $d$ controls the oscillator’s peak frequency deviation from $\omega_c$, yielding instantaneous frequency

$$\omega_{i,\text{fm}}(t) = \omega_c + d \cos(\omega_m t), \quad (12)$$

and employing the relationships given in (7), instantaneous phase

$$\theta_{i,\text{fm}}(t) = \int_0^t \omega_i(t) \, dt = \int_0^t \omega_c \, dt + d \int_0^t \cos(\omega_m t) \, dt = \omega_c t + \frac{d}{\omega_m} \sin(\omega_m t) + \phi_c. \quad (13)$$

Expression (13) shows that FM may be equivalently expressed as phase modulation (PM) where it is the initial phase from (1) that is made sinusoidally time varying

$$\phi(t) = I \sin(\omega_m t) + \phi_c, \quad (14)$$

where the amplitude of the phase modulating sinusoid,

$$I = \frac{d}{\omega_m}, \quad (15)$$

is known as the index of modulation for its influence on the frequency bandwidth of the signal—the extent to which sidebands that are harmonics of $f_m$ spread from $f_c$. Note that (15) shows a factor of $\omega_m$ between FM’s peak frequency deviation parameter $d$ and PM’s index of modulation parameter $I$.

3.1 FM and PM representations

FM synthesis is often implemented using its theoretically equivalent (and numerically advantageous) PM form which, for the generation of a real signal, may be expressed as

$$x_c(t) = \cos(\omega_c t + I \sin(\omega_m t)), \quad (16)$$

or equivalently as the real part of a complex exponential signal of the form described in Section 2

$$x_c(t) = \Re\{z_c(t)\} = \Re\{e^{j(\omega_c t + \Im\{z_m(t)\})}\}, \quad (17)$$

where the time-varying initial phase,

$$\Im\{z_m(t)\} = \Im\{Ie^{j\omega_m t}\} = I \sin(\omega_m t), \quad (18)$$

is the imaginary part of the complex exponential modulating sinusoid $z_m(t) = I e^{j\omega_m t}$.

Expressing the complex PM oscillator in discrete time (by the substitution $t \rightarrow nT$) at sample $n - 1$:

$$z_c(n-1) = I e^{j(\omega_c(n-1)T + \Im\{z_m(n-1)\})}, \quad (19)$$

then adding and subtracting $\Im\{z_m(n)\}$ to its argument:

$$j(\omega_c nT + \Im\{z_m(n)\} - \omega_c T - \Im\{z_m(n) - z_m(n-1)\}), \quad (20)$$
shows (19) can be represented as the rotation
\[ z_c(n - 1) = z_c(n)e^{j(-\omega_c T - \Im\{z_m(n) - z_m(n - 1)\})}, \quad (21) \]
or in causal form as a rotation of its previous sample, also known as the FM representation (because the rotation angle is the instantaneous frequency):
\[ z_c(n) = e^{j(\omega_c T + \Im\{z_m(n) - z_m(n - 1)\})}z_c(n - 1), \quad (22) \]
where the modulator signal may be similarly expressed by the rotation \( z_m(n) = e^{j\omega_m T}z_m(n - 1) \). Notice the rotation angle given in (22), the instantaneous frequency of the oscillator \( \omega_i(n) = \omega_c T + \Im\{z_m(n) - z_m(n - 1)\} \), is equivalent to a finite difference approximation of the derivative of the continuous-time instantaneous phase given in (17).

4. Loopback FM

Here we consider an FM system whereby the carrier signal is looped back with a feedback coefficient so that it modulates its own frequency \( f_c \). The result is loopback FM, so called because the term feedback FM has been previously coined \([2]\) for a similar (but distinct) technique whereby the carrier signal is fed back to modulate its own initial phase, creating a change in the spectrum that has been used for the synthesis of musical tones. In \([3]\), for example, genetic algorithms are used to find optimal parameters for nested modulator and feedback FM synthesis of trumpet tones, a tenor voice and the sound produced by the Chinese pipa. Unlike feedback FM, which preserves the pitch but changes the spectrum, loopback FM produces nonlinear modal coupling, a process known to create pitch glides (typical of feedback systems and desirable to simulate in dynamic acoustic models). It is this change in sounding frequency (pitch glides) that we explore herein, ultimately revealing expressions as a function of FM/PM parameters \( d, I \), and \( f_c \) in Section 5.

The complex oscillator \( z \) may be coupled to itself by adding a portion (indicated by coupling coefficient \( B \)) of its real and/or imaginary part to its rotation angle \( \omega_c T \):
\[ \omega_i = \omega_c T + B\omega_c T\Re\{z\}, \quad (23) \]
with factor \( \omega_c \) being due to the relationship between FM/PM parameters \( d \) and \( I \) as shown in (15). Following the discrete-time FM equation in (22), this produces the loopback FM equation for time sample \( n \):
\[ z(n) = e^{j(\omega_c T + B\omega_c T\Re\{z(n - 1)\})}z(n - 1), \quad (24) \]
where a unit sample delay is necessarily introduced for implementation and where \( z(0) \) may be set to 1 to excite the system (or obtain its impulse response). The instantaneous frequency (with unit-sample delay) of (24) is
\[ \omega_i(n) = \omega_c T + B\omega_c T\Re\{z(n - 1)\}. \quad (25) \]
and the corresponding instantaneous phase, found by integrating (25) with respect to \( n \), is
\[ \theta_i(n) = \int_0^n \omega_i(n) \, dn = \omega_c nT + B\omega_c T\Re\left\{\int_0^n z(n - 1) \, dn\right\}. \quad (26) \]
Expression (26) produces an expression involving an integral that would have to be solved in order to represent as PM \( z(n) = e^{j \theta_i(n)} \). One possible implementation being via numerical integration:

\[
\theta_i(n) = \omega_c n T + B \omega_c T \Re \left\{ \sum_{k=0}^{n-1} z(k) \right\},
\]

(27)

where \( e^{j \theta_i(n)} \) may be shown to be exactly equivalent to (24), the real part of which is shown in Figure 1. In addition to the introduced sample delay and numerical inaccuracies based on sampling rate, this solution doesn’t give further insight into the question of the pitch (frequency) at which \( \Re\{z\} \) will sound. In the following sections we therefore find analytical expressions for \( \theta_i \) and \( z \).

Figure 1: The real part of \( z(n) \), plotted with respect to sample index, for a carrier frequency of \( \omega_c = 2\pi f_c \) where \( f_c = 700 \) Hz and feedback coefficient of \( B = .95 \).

## 5. Analytic Solution for Phase \( \theta_i(n) \) and the Stretched Allpass Filter

As shown in Figure 1, the real part of the loopback FM equation given in (24) is a periodic function having a period of \( M \) samples and a sounding frequency of \( f_0 = f_s/M \) Hz, a signal also described by the real part of the closed form representation

\[
H(n) = \frac{b_0 + e^{j \omega_0 n T}}{1 + b_0 e^{j \omega_0 n T}},
\]

(28)

where \( \omega_0 = 2\pi f_s/M \). The angle of (28) is given by

\[
\angle H = \omega_0 n T - 2 \tan^{-1} \left( \frac{b_0 \sin(\omega_0 n T)}{1 + b_0 \cos(\omega_0 n T)} \right),
\]

(29)

which may be made equal to the angle (instantaneous phase) of \( z \) given in (26). Expression (28) is similar to the “stretched” allpass filter transfer function [4], though here it is used as a time-domain signal that is a function of time sample \( n \). Its use in this work is also reminiscent of the second-order power-preserving allpass filters for frequency modulation in [5].
The integral in (26) can now be expressed analytically by finding the integral of $H$. Since this is a continuous-time operation, substitution $nT \rightarrow t$ is made before integrating $H$ with respect to $t$:

$$ \int_0^t H(t) \, dt = \int_0^t \frac{b_0}{1 + b_0 e^{j\omega_0 t}} \, dt + \int_0^t \frac{e^{j\omega_0 t}}{1 + b_0 e^{j\omega_0 t}} \, dt $$

(30)

which, after some calculus beyond what is permitted here, reduces to

$$ \int_0^t H(t) \, dt = b_0 t + \frac{(1 - b_0^2) \log(1 + b_0 e^{j\omega_0 t})}{j\omega_0 b_0} $$

(31)

for static $\omega_0$ and $b_0$. Expressing the logarithmic term in (31) as a sum of real and imaginary parts,

$$ \log(1 + b_0 e^{j\omega_0 t}) = \log(A) + j\nu $$

(32)

where

$$ A = \sqrt{1 + 2b_0 \cos(\omega_0 t) + b_0^2} $$

(33)

and

$$ \nu = \tan^{-1} \left( \frac{b_0 \sin(\omega_0 t)}{1 + b_0 \cos(\omega_0 t)} \right) $$

(34)

greatly simplifies the expression for the real part of (31) for static $b_0$ and $\omega_0$,

$$ \Re \left\{ \int_0^t H(t) \, dt \right\} = b_0 t + \frac{1 - b_0^2}{j\omega_0 b_0} \nu $$

(35)

Returning to discrete time with the substitution $t \rightarrow nT$, (35) may be used as the solution to the integral term in (26) with the final analytic solution to the phase (angle) of $z$ given by

$$ \theta_i(n) = \omega_c nT + B\omega_c \Re \left\{ \int_0^n z(n) \, dn \right\} = \omega_c nT(1 + Bb_0) + B\omega_c \frac{1 - b_0^2}{\omega_0 b_0} \nu $$

(36)

for yet unknown values $b_0$ and $\omega_0$.

### 5.1 Solving for static $b_0$ and $\omega_0$

Expressions for $b_0$ and $\omega_0$ may be obtained by first making $\theta_i(n)$ in (36) equal to $\angle H$ given in (29):

$$ \omega_c nT(1 + Bb_0) + B\omega_c \frac{1 - b_0^2}{\omega_0 b_0} \nu = \omega_0 nT - 2 \nu $$

(37)

Making linear terms (those not a product of oscillatory $\nu$) on both sides of (37) equal yields one expression for $\omega_0$:

$$ \omega_0 = \omega_c (1 + Bb_0) $$

(38)

and, making oscillatory terms equal on both sides of (37) yields a second expression for $\omega_0$:

$$ \omega_0 = \frac{B \omega_c (1 - b_0^2)}{-2b_0} $$

(39)

where equating expressions for $\omega_0$ (38) and (39) yields the quadratic equation

$$ Bb_0^2 + 2b_0 + B = 0 $$

(40)
and a solution for $b_0$ given by
\[ b_0 = \pm \sqrt{1 - B^2} - 1. \] (41)

Finally, substituting (41) into (38) yields an expression for instantaneous frequency $\omega_0$ as a function of $B$ and $\omega_c$:
\[ \omega_0 = \omega_c \left( 1 + B \frac{\pm \sqrt{1 - B^2} - 1}{B} \right) = \pm \omega_c \sqrt{1 - B^2}. \] (42)

5.2 Time variation—pitch glides

The derivation in the previous section assumes a static loopback variable $B$ which, by (42), also produces a static frequency $\omega_0$. To accommodate for the possibility of time variation, the “stretched” allpass expression in (28) is made more generalized:
\[ H(n) = \frac{b_0 + e^{j\theta_H(n)}}{1 + b_0 e^{j\theta_H(n)}}, \] (43)

where $\theta_H$ is given by the integral of $\omega_0$ given in (42). If $B$ is changed exponentially (as it would be if $z$ had an exponential decay in its amplitude),
\[ B(n) = g^n, \text{ for } g < 1, \] (44)

then $\theta_H$ is given by the integral
\[ \theta_H(n) = \int_0^n \omega_0(n) = \pm \int_0^n \omega_c \sqrt{1 - g^{2n}} \, dn. \] (45)

Using $u$-substitution, where
\[ u = \sqrt{1 - g^{2n}} \quad \text{and} \quad \frac{du}{dn} = -\frac{1}{2g^{2n}} g^{2n} \log(g), \] (46)

an expression is obtained for $dn$,
\[ dn = -\frac{1 - g^{2n}}{g^{2n} \log(g)} \, du = \frac{u}{\log(g)(u^2 - 1)} \, du, \] (47)

and (45) may be expressed as
\[ \theta_H(n) = \pm \int_0^n \omega_c \sqrt{1 - g^{2n}} \, dn = \frac{\omega_c}{\log(g)} \int_0^u \frac{u^2}{u^2 - 1} \, du. \] (48)

Solving the integral in (48) yields the final expression for $\theta_H(n)$ given by
\[ \theta_H(n) = \frac{\omega_c}{\log(g)} \left( u - \tanh^{-1}(u) + C \right), \] (49)

where $u$ is defined in (46). Figure 2 shows the real part of $H$ for $\theta_H$ given in (49), with a solid darkened curve plotting time-varying sounding frequency
\[ f_0(n) = f_c \sqrt{1 - B(n)^2}. \] (50)

The close agreement between spectrum and curve shows that $H$ produces the pitch glide predicted by (50). Finally, it should be noted that $\angle H$ may be used as the instantaneous phase in a PM representation of the system.
In this work we examine the pitch glide that occurs in a loopback FM configuration, where the oscillator is looped back to modulate its own frequency, in a nonlinear self-coupling that is known to alter pitch. When the loopback FM coefficient is made time varying, pitch glides, result.

By reviewing the relationship between frequency and phase, and FM and PM representations, we expose the need to analytically solve an integral in order to obtain a more numerically accurate delay-free expression for the phase. By making the observation that the system is equal to a stretched allpass function $H$, an analytic solution to the phase is made possible and, because the phase of $H$ is known, stretched allpass parameters $b_0$ and $\omega_0$ may be mapped to loopback FM parameters $B$ and $\omega_c$. Finally, a more generalized expression for the stretched allpass function is given, which involves integrating sounding frequency $f_0$ (when it is made time varying), providing loopback FM parameters for specified pitch trajectories.

REFERENCES