Identifying vibration sources on structures is frequently impossible to perform via direct measurements, and has to be carried out by solving an inverse problem. In this context, this article presents an extension of the force analysis technique. This extension allows identifying transient loads on plate-like structures from the measurement of transverse displacements on a grid of points. The proposed method consists in injecting this data into the discretized equation of motion expressed in time and space domains to get the residual force of the equilibrium. To overcome the sensitivity of the problem to noise, a regularization step using Tikhonov’s $\ell_2$ cost function is performed. This formulation allows inspecting only part of the structure, and finding the time signature of dynamic loads. By adding prior knowledge on the source’s location, the proposed method also allows pinpointing material inhomogeneities or defects, which cause non-zero residual forces outside the loading area. The technique is experimentally validated on a plexiglass plate on which an added mass representing a material defect is placed. The impact force is recovered from measurements on the plate: its location is successfully identified and the impact hammer data is compared to the reconstructed force. The knowledge of the force location is used to successfully detect the added mass.

Keywords: transient force analysis technique, impact

1. Introduction

In many competitive sectors such as aeronautics, naval or building industry, new materials have been widely used in the few past decades to design light and thin structures. However, these structures can be damaged by vibration. This results in a critical need to estimate and control vibration levels of the designed parts. Due to the increasing processing power of modern computers, quantifying displacement fields caused by a given excitation can be performed accurately and quickly. However, sources often remain unknown because they cannot be directly measured, and therefore, have to be estimated. Since the 1970’s various methods such as Transfer Path Analysis [1], techniques based on structural intensity [2] or other global methods [3] were developed to estimate vibration sources. The Virtual Fields Method using the principle of virtual work was developed since the late 80’s, and recently adapted to perform stationary [4] or time-resolved [5] force reconstruction. In parallel, Pezerat et al. proposed the Force Analysis Technique (FAT), based on a local formulation of the equation of motion, which does not require knowing the boundary conditions of the problem. This method was first developed for stationary sources applied...
on simple structures such as beams [6] and plates [7], and was further extended to arbitrarily shaped structures by using a local finite element operator [8]. From this formalism, a significant number of advances were made to reduce errors due to the discretization scheme [9], to improve regularization [10], to estimate the physical parameters of a plate [11], or to detect material inhomogeneities [12]. A similar approach was derived by Bucaro et al. [13] to pinpoint defects knowing the stationary displacement field and the wave equation of the structure.

The present work aims at adapting the Force Analysis Technique to identify transient loads applied on thin plates. The time-domain formulation developed for the present method is an alternative to the time-domain reconstruction from frequency-domain formulations, that shows some limitations, especially in the low-frequency range [14]. In section 2, theoretical aspects behind transient FAT are explained. In section 3, the proposed method is applied on a plexiglass plate to recover the main characteristics of the excitation and to localize a structural defect.

2. Theory of transient FAT

In this section, the theoretical framework of this study is set. The equation of motion is presented, and its discretized counterpart is given. An efficient implementation scheme using FFT is proposed.

2.1 Theoretical framework

Let us consider a plate of thickness $h$, subjected to a transient normal force. In order to reduce the complex 3D vibration problem to a 2D problem, Kirchhoff’s hypotheses for small deformations are assumed:

1. no shear deformation in the plate,
2. no elongation in the transverse direction,
3. rotational inertia negligible.

From these hypotheses, one can derive the equation of motion using the principle of virtual works:

$$\frac{Eh^3}{12(1-\nu^2)} \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 y^2} + \frac{\partial^4 w}{\partial x^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = f \quad (1)$$

where $w(x, y, t)$ and $f(x, y, t)$ are respectively the transverse displacement and the normal surfacic load acting on the plate at location $(x, y)$ and time $t$, $E$ is the Young’s modulus, $\nu$ is the Poisson’s ratio, and $\rho$ is the density of the plate.

The main idea behind FAT is to feed the displacement data into the left-hand side of Eq. (1) to get the force distribution as a function of time and space. However, since the input data is measured from an actual structure, the derivatives have to be approximated numerically using appropriate finite difference schemes.

2.2 Discretized formulation

In the following, we assume that the displacement field $w(x, y, t)$ is regularly sampled in all directions, with spacings $\Delta_x, \Delta_y, \Delta_t$. Let us denote each sample of the displacement field by $w_{i,j,n}$, where $i, j,$ and $n$ correspond respectively to the index of the sample in the $x, y,$ and $t$ dimension. The approximation of the space derivatives are given by the following centered finite difference schemes:

$$\frac{\partial^4 w}{\partial x^4} \approx \delta_{i,j,n}^x \left( w_{i-2,j,n} - 4w_{i-1,j,n} + 6w_{i,j,n} - 4w_{i+1,j,n} + w_{i+2,j,n} \right) \quad (2)$$
\[
\frac{\partial^4 w}{\partial y^4} \approx \delta_{i,j,n} = \frac{1}{\Delta y} \left( w_{i,j-2,n} - 4w_{i,j-1,n} + 6w_{i,j,n} - 4w_{i,j+1,n} + w_{i,j+2,n} \right) 
\]
(3)

\[
\frac{\partial^4 w}{\partial x^2 \partial y^2} \approx \delta_{i,j,n} = \frac{1}{\Delta x^2 \Delta y} \left( w_{i-1,j-1,n} - 2w_{i,j-1,n} + w_{i+1,j-1,n} - 2w_{i-1,j,n} + 4w_{i,j,n} - 2w_{i+1,j,n} + w_{i-1,j+1,n} - 2w_{i,j+1,n} + w_{i+1,j+1,n} \right) 
\]
(4)

The time derivative is also approximated with a second-order centred scheme:

\[
\frac{\partial^2 w}{\partial t^2} \approx \delta_{i,j,n} = \frac{1}{12\Delta t} \left( -w_{i,j,n-2} + 16w_{i,j,n-1} - 30w_{i,j,n} + 16w_{i,j,n+1} - w_{i,j,n+2} \right) 
\]
(5)

Since this paper presents an extension of the classical FAT method for transient excitations, the schemes presented in Eqs. (2), (3), and (4) were chosen to be the same as the ones used in classical FAT [7]. To be consistent with these previous developments, Eq. (5) is also chosen to be centred and of second order.

By injecting Eqs. (2) to (5) into the equation of motion, we get:

\[
\frac{Eh^3}{12(1-\nu^2)} \left( \delta_{i,j,n}^4 + 2 \delta_{i,j,n}^{2y} + \delta_{i,j,n}^{4y} \right) + \rho h \delta_{i,j,n}^{t2} = f_{i,j,n} 
\]
(6)

According to Eq. (6), the force acting on the structure at a given sample \((i, j, n)\) can be computed from the samples of \(w\) involved in Eqs. (2) to (5). In other words, the mechanical loading can be expressed as a convolution of the three-dimensional matrix \((w_{i,j,n})\) with a kernel \(k = (k_{i,j,n})\). The \(k_{i,j,n}\) can be derived from Eq. (6): they are directly determined by the coefficients of the proposed discretization schemes, and the physical properties of the plate (Young’s modulus, density, thickness, and Poisson’s ratio). This 3D-convolution is a linear operation, therefore, it can be represented by a matrix \(C_k\), which is block-circulant:

\[
C_k w = f 
\]
(7)

A simple computation of this convolution yields erroneous results the moment the input data is corrupted by noise. This is due to the fact that, although Eq. (7) does not have to be inverted (since the aim is to compute \(f\) from the measured \(w\)), it has the characteristics of an inverse problem. Physically, this kind of difficulty is encountered when trying to infer the causes of a phenomenon (here, the excitation force) from the observation of its consequences (here, the vibration field). This circulant formalism is therefore unable to handle the ill-posedness of the problem, and has to be supplemented with regularization.

### 2.3 Regularization and windowing steps

The proposed approach used to solve this ill-posed problem is to perform a regularization. In order to apply Tikhonov’s regularization [15], the inverse of the system (7) has to be considered. The regularized solution \(f_r\) is computed by adding a penalty term on the norm of the solution to the initial problem:

\[
f_r = \arg \min_f \left( ||C_k^{-1}f - w||^2 + \lambda ||f||^2 \right),
\]
(8)

where \(|| \cdot ||\) stands for the \(\ell_2\) norm, and the regularization parameter \(\lambda\) is a scalar. This parameter sets the balance on the trade-off between the residue of the system and the energy of the excitation force. Therefore, it must be chosen carefully since the regularized solution \(f_r\) strongly depends on this term.
Several methods are frequently used to choose automatically this parameter by optimizing a certain cost function. This is for example the case for the L-curve method [16], Generalized Cross-Validation [17], and empirical Bayesian regularization [10]. An explicit solution to Eq. (8) exists, and can be handily expressed by considering the diagonalization of $C_k$. Since $C_k$ is block-circulant, it can be diagonalized using the 3D-DFT matrix $F$ [18]:

$$C_k = F^{-1} \Delta F,$$

(9)

where $\Delta$ is a diagonal matrix formed by the Fourier coefficients of $k$. Using this decomposition, an explicit solution to Eq. (8) exists, and can be arranged as follows:

$$f_r = \left( I_N + \lambda F^{-1} \Delta^H F \right)^{-1} C_kw,$$

(10)

where $I_N$ stands for the identity matrix of size $N \times N$ ($N$ is the total number of elements of the domain $N_x \times N_y \times N_t$), and the superscript $H$ stands for the conjugate transpose. This regularized solution can be looked at as the naive solution $C_kw$ on which a filter is applied. This expression can be further simplified using the 3D-Fourier transform (denoted by $F$) of $w$ and $k$:

$$f_r = F^{-1} \left[ \frac{1}{1 + \lambda |F(k)|^2} F(k) \cdot F(w) \right].$$

(11)

This expression allows efficient computations, and avoids building explicitly the $N \times N$ matrix $C_k$, which can be large. In addition to this regularization, a windowing step is necessary to avoid the edge effects due to the convolution. Indeed, the previous formulation relies on circular convolution, which assumes periodicity of the signals convolved, therefore, on the edges of the measurement field, where the non-zero elements of the kernel overlaps with the periodic repetition of the domain, undesired components are introduced. They must be removed by multiplying the force with a 3D-Tukey window $T(x, y, t)$ before the filtering step:

$$T(x, y, t) = T_{\alpha_x} \left( \frac{x}{L_x} \right) T_{\alpha_y} \left( \frac{y}{L_y} \right) T_{\alpha_t} \left( \frac{t}{L_t} \right),$$

(12)

The windowing parameters $\alpha_x$, $\alpha_y$ and $\alpha_t$ define the width of the decaying part of $T$ (of width $\alpha_x N_x$, $\alpha_y N_y$, and $\alpha_t N_t$, respectively in directions $x$, $y$, $t$). A simple criteria can be applied to choose them: the tapered part should at least cover the zone where the overlap occurs (of width 5 points in all directions, since Eq. (6) involves samples from $w_{i-2,j,n}$ to $w_{i+2,j,n}$, from $w_{i,j-2,n}$ to $w_{i,j+2,n}$, from $w_{i,j,n-2}$ to $w_{i,j,n+2}$). In the following, this rule of thumb is applied, and the windowing coefficients are set to $\alpha_x = 2 \times \frac{5}{N_x}$, $\alpha_y = 2 \times \frac{5}{N_y}$, $\alpha_t = 2 \times \frac{5}{N_t}$. The expression derived in Eq. (11) and the proposed windowing are implemented in the procedure summed up in Fig. 1. The convolution kernel must be built and zero-padded to match de size of the displacement field. The convolution is made by multiplying signals in the Fourier domain. The windowing is applied before applying the regularization filter.

3. Experiments

In this section, an experimental validation of transient FAT is achieved. After presenting the test bench and the measurement procedure, the reconstructed force is analyzed to identify the impact on the plate (location and time signature), and to detect a defect in the plate.

3.1 Experimental setup

The structure under investigation is a rectangular plexiglass plate of dimension $50 \times 30$ cm, and of thickness 6.5 mm, on which a mass of 23 g is glued at position $(0.34, 0.10)$ m ($(0, 0)$ m is the upper left
corner of the plate). The plate is hooked on a rigid base frame. The Young’s modulus of the plate is $E = 3.1$ GPa, the density is $\rho = 1140$ kg.m$^{-3}$, and the Poisson’s ratio is $\nu = 0.37$. An impact hammer (PCB 086C01) is mounted on ball bearing, and acts as a pendulum hitting the plate at position $(0.21, 0.18)$ m. The hammer is released by an electromagnet to ensure satisfying repeatability of the shock (see Fig. 2).

A vibrometer (Polytec OFV-505/5000) is used to scan the transverse velocity of the full plate, over a 1 x 1 cm grid. The data is sampled at 102.4 KHz, and the duration of acquisition is 20 ms. The data obtained from the impact hammer is stored and used to recover the phase of the normal displacement at each point. Since the laser vibrometer outputs the normal velocity, a numerical integration is necessary to get the normal displacement of the plate, otherwise, the time derivative of the loading $\frac{\partial P}{\partial t}$ would be computed instead of $f_r$. This step is performed via cumulative trapezoidal numerical integration. Even if it was not the case in our study, measurement noise can produce drift during integration. The authors suggest removing it by high-pass filtering the signal before applying transient FAT.

![Figure 1: Block diagram summarizing the steps involved to compute the force distribution. $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes 3D-FFT and inverse 3D-FFT. The hatched area represents the samples of the output affected by the windowing step.](image)

![Figure 2: Front view (a) and side view of the setup (b). 1) base frame. 2) plexiglass plate. 3) laser vibrometer. 4) electromagnet. 5) impact hammer. 6) added mass.](image)
3.2 Results and analysis

(a) Displacement map 1.5 ms after the impact. (b) Displacement time signals at 3 positions.

Figure 3: Measured displacement field.

A snapshot of the measured displacement field is presented in Fig. 3a. The spatial smoothness of the data indicates that signals acquired at each point are successfully phased, and that the effect of drift due to time integration can be neglected. From unprocessed displacement maps, the position of the impact cannot be precisely determined, and the added mass is not detectable. Fig. 3b also shows that the time evolution of the impact force is hard to infer from raw data, and that signals obtained at the impact and mass positions do not have special features compared to displacements at other positions of the plate.

The method described in Fig. 1 is then applied on the measured dataset. The regularization parameter $\lambda = 10^{-21.1}$ is selected using Generalized Cross Validation. After processing the displacement field, the loading $f_r$, which is presented on Fig. 4 is known as a function of time and space. Between $t = 0$ ms and $t = 0.7$ ms, a peak located at $(x, y) = (0.21, 0.18)$ m indicates the estimated position and duration of the excitation. It is visible on the root mean square (RMS) map in Fig. 4a. The cross denotes the position of the impact, and the circle denotes the position of the mass.

After the impact, the plate is not subjected to any active load. However, a non null residual is observed at the position of the added mass. This mass does not affect the stiffness term in Eq. (6); in the inertia term however, the locally added mass is not taken into account. Therefore, in order for the equation to be balanced, the right-hand term of Eq. (6) is non-null even though no force is applied at this position. In Fig. 4b, the root-mean square value of the field is computed between $t = 2.5$ and $3.5$ ms. This disturbed equilibrium is clearly visible and matches the position of the defect.

(a) During the impact ($0 \leq t \leq 0.7$ ms). (b) After the impact ($2.5 \leq t \leq 3.5$ ms).

Figure 4: RMS values of the distributed loading computed using transient FAT.
In addition to impact localization and fault detection, the proposed method allows assessing the time signal of the impact force by integration of the surface load over the impact region, as shown in Fig. 5. The signals have been normalized prior to comparison. In fact, the reconstructed force has a significantly lower norm than the actual signal measured with the impact hammer: due to the filtering made during the regularization step, important components of the signal are removed, as well as undesired components that arise from the amplification of the measurement noise. Therefore, the output of the proposed method does not provide accurate information about the intensity of the excitation force. However, even though the duration of the contact is overestimated, the instant of the impact is correctly identified, and the overall shape of the signal is reconstructed.

Figure 5: Superposition of the impact signal measured by the impact hammer and the force obtained by transient FAT.

4. Conclusion

This paper presented a method for the identification of transient loads applied on a plate, from the knowledge of displacements and material properties of the structure. It was shown that Tikhonov’s $\ell_2$ regularization addresses the ill-posedness of the problem, and that the use of a circulant formalism allows efficient computation of the regularized solution by using three-dimensional Fast Fourier Transform. The proposed method was successfully applied on data obtained from a plexiglass plate, to demonstrate that time-resolved identification and localization of transient loads can be performed, even though accurate estimation of magnitude is not possible. An added mass bonded on the plate was successfully detected and localized, which provides a proof of concept for fault detection on plate-like structures. Future works could include the adaptation of the proposed method for non-isotropic materials, or could improve transient FAT by taking into account damping terms in the equation of motion.

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