ON CONTINUUM MODELLING OF WAVE PROPAGATION IN HEXAGONAL LATTICES.

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Wave propagation in hexagonal lattices is characterised by a transition from isotropic to anisotropic regime as the frequency increases. This feature cannot be observed when the lattice material is modelled within the framework of classic Cauchy theory of elasticity, except when the real geometry of the microstructure is explicitly described. Homogenized equivalent continua can capture the onset of this anisotropy only if tensors involved in the constitutive law are at least of order six. This requirement is met in the case of Strain-Gradient Elasticity (SGE). In this work the SGE model is calibrated to quantitatively describe wave propagation within hexagonal lattices in a sufficiently large region of the dispersion diagram, by fitting the dispersion relations obtained from a Bloch-Floquet analysis on the unit cell.

Strain-gradient elasticity, anisotropy, Higher-order tensors, Beam steering, Wave propagation

1. Introduction and motivations

The study of wave propagation within periodic architectured materials is a topic of increasing interest. This subject finds its origin in the field of electromagnetism, where it drove the development of innovative materials and devices, e. g. smart wave guides or cloaking devices. Indeed, materials with exotic properties (e. g. stop bands, energy focusing) are obtained by exploiting the periodic nature of such materials. The same concept can be successfully applied to elastic waves for designing materials capable of changing the direction of propagation of the energy (e. g. wave beaming), to enhance the non-destructive characterization properties of the material itself (e. g. materials with a specific acoustic signature when damaged). The behaviour of waves propagating in these media strongly depends on frequency. For example some of them have an isotropic behaviour at low frequencies and become anisotropic at high frequencies. Well known is the case of hexagonal lattices, used in so-called honeycomb structures, for which an isotropic (in 2D) or a transverse isotropic (in 3D) model is commonly used. However, when performing a simple wave propagation test, it can be easily observed that a breaking of symmetry occurs
when frequency of the source is sufficiently high. This effect can be seen in Fig. 1, where the simulation is conducted on the real honeycomb geometry (classic solid mechanics equations in 2D plane strain). A shear pulse is applied in the center of the domain, and the total displacement is plotted.

![Snapshots of the propagation of a shear wave at low and high frequency. Full field honeycomb model (classic solid mechanics in 2D plane strain). The norm of the displacement is displayed, arbitrary units are used. Image extracted from [1].](image)

**Figure 1**: Snapshots of the propagation of an in plane shear wave at low and high frequency. Full field honeycomb model (classic solid mechanics in 2D plane strain). The norm of the displacement is displayed, arbitrary units are used. Image extracted from [1].

### 2. Main results

In order to capture the onset of the anisotropic transition in the framework of continuum mechanics, i.e., without describing the geometry of the unit cell, the classic Cauchy model is not sufficiently rich and a generalization must be sought. A guideline for choosing the correct generalized model is provided by the Neumann principle, which states that

*if a crystal is invariant with respect to certain symmetry operations, any of its physical properties must also be invariant with respect to the same symmetry operations.*

In order to translate this into requirements for the tensors appearing in the constitutive laws, we can cite the Hermann theorem from crystal physics, which states that

*if we consider an r-rank tensor with reference to a material having a N-fold axis of symmetry, and r<N, then this tensor property affectively conforms to an infinitely-fold symmetry axis parallel to the N-fold axis.*

The Hermann theorem provides a sufficient condition for transverse anisotropy. Following this theorem, it is clear that, since a honeycomb crystal has a 6-fold axis of symmetry, a constitutive tensor of rank 6 is needed to capture the anisotropy. As anticipated, this excludes the fourth order elastic tensor. As it will be shown hereafter, this is the case of Strain Gradient Elasticity (in this work we will refer to the case of Mindlin’s Type II [2] strain gradient elasticity theory). More details on the model, the constitutive law and the identification procedure for the coefficients can be found in [2, 3, 4, 5].

In order to give a quick overview of the strain gradient elasticity model, the basic equations will be recalled here, starting from bulk equation:

\[
\nabla \cdot \left( \sigma - \nabla \cdot \tau \right) = p - \nabla \cdot q
\]

(1)
where $\sigma$ is the Cauchy stress tensor, $\tau$ is hyperstress tensor, $p$ is the momentum vector and $q$ is the hypermomentum tensor. For completeness, in this work tensors of order ranking from 0 to 6 are denoted, respectively, by $a$, $a$, $a$, $a$, $a$, $a$ and $a$. In the case of a centrosymmetric material, and it is the case of 2D honeycomb lattices, constitutive equations will have the following structure:

$$
\begin{pmatrix}
\rho \\
p \\
0 \\
q \\
\sigma \\
\tau
\end{pmatrix} = 
\begin{pmatrix}
\rho I & 0 & 0 & 0 \\
0 & J & 0 & 0 \\
0 & 0 & C & 0 \\
0 & 0 & 0 & A \\
0 & 0 & 0 & A
\end{pmatrix}
\begin{pmatrix}
\nabla v \\
\nabla \epsilon \\
\nabla \epsilon \\
\nabla \epsilon \\
\nabla \epsilon
\end{pmatrix}
$$

where $\rho$ is the macroscopic mass density, $v$ the velocity, $\epsilon$ the infinitesimal strain tensor, $I$ the second order identity tensor, $J$ micro inertia tensor, $C$ is the classical elasticity tensor, $A$ the sixth-order elasticity tensor. The sixth order tensor is then sufficiently rich to capture the onset of the symmetry breaking when increasing the frequency. Once the model correctly calibrated (the identification procedure is based on Floquet-Bloch analysis, more details in [5, 1]), a simulation performed with the same input as for the full field model has been conducted using the strain gradient elasticity model. The results are in Fig. 2. As it can be appreciated, the model is capable of predicting the correct wave propagation. More details on possible applications involving wave beaming can be found in [1].

![Figure 2: Snapshots of the propagation of an in plane shear wave at low and high frequency. Strain Gradient Elasticity model in 2D. The norm of the displacement is displayed, arbitrary units are used. Image extracted from [1].](image)

**REFERENCES**


