This work studies the estimation of material properties in complex systems subjected to dynamic coupling and exhibiting a resonant behaviour. The model parameter search can be conveniently formulated as an inverse estimation procedure consisting in minimising a distance between predicted and measured responses. However, classical optimisation algorithms face the existence of local minima, which are due to resonances and geometrical symmetries in the system of interest. An incremental approach is here presented, consisting of an estimation of the model parameters using a sequence of sub-problems. The starting sub-problem is chosen as a limiting case where the system exhibits an asymptotic, non-resonant behaviour, sensitive to a subset of the unknown model parameters. Subsequent sub-problems gradually incorporate the full complexity of the problem and its sensitivity to the complete set of model parameters. As a result, the inverse estimation procedure is guided to the true solution of the problem, while avoiding local minima. The approach is applied to the estimation of geometrical and material properties of systems including expansion chambers and porous media, using impedance tube measurements as the experimental target. In order to examine the methodology and its inherent features, two inversion approaches are applied, namely a deterministic framework using a gradient-based optimiser and a Bayesian framework using a Markov chain Monte Carlo method. In particular, the sensitivity of the solution to variations in the model parameters is explored at the various stages of the step-wise procedure.

Keywords: parameter estimation, resonant systems, impedance tube, incremental approach

1 Introduction

The estimation of geometrical and material properties of systems and structures is at the core of acoustical and mechanical metrology. Most of such problems, including those involving linear systems, exhibit a non-linear relationship between their physical properties and the system’s response. Therefore, the estimation of such properties is often achieved in an inverse manner by means of iterative methods aiming at minimising the difference between an observed response and a predicted response, in the space of unknown properties. As such, inverse problems for parameter estimation in dynamic problems are subjected to the existence of local minima, thereby hindering the retrieval of the true parameter values.
This apparent non-uniqueness of the solution arises as a consequence of resonances or symmetries, common in dynamic coupled problems. The underlying challenge lies in the self-similarity of the problem at different frequencies or at different values of its properties [1, 2].

An incremental method is here proposed, consisting in solving successive sub-problems over a gradually increasing frequency range, typically starting from a low-frequency range where the system behaviour is asymptotic, until reaching the full frequency range. Such a method has been successfully applied to coupled resonant vibroacoustic problems [3, 4]. The authors have observed that the method guides the unknowns of the problem towards the true parameter values. The present paper extends the application of the method to the estimation of the properties of a system composed of an expansion chamber coupled to a rigidly backed sample of poroelastic material, from the knowledge of the frequency-dependent total impedance. The example treated in this paper is fully simulated.

In the present paper we present two different model inversion techniques for solving the sub-problems. The first technique consists of a gradient-based optimisation routine, allowing for a reduced number of model evaluations, particularly relevant in cases where the model relies on a finite element simulation [2]. The second technique is a Markov chain Monte Carlo method (MCMC) using the Metropolis-Hastings algorithm, which requires a larger number of model evaluations, but provides detailed insight into the solution in a probabilistic sense. This includes variable point estimates and uncertainty, as well as the sensitivity of the estimation with respect to variations in the unknowns.

The paper is organised as follows. Section 2 presents the incremental-frequency method, Sec. 3 describes the deterministic and statistical model inversion tools and Sec. 4 presents the application to the parameter estimation of a coupled expansion chamber and poroelastic material.

2 Method

2.1 Observation model

The retrieval of geometrical or material parameters is here based on the observation of a frequency response, in the form

\[ g_0(\omega_m), \quad m \in [1, M], \]

(1)

where \( \omega_m \) is a sequence of discrete circular frequencies at which the observation is available. The system is assumed to behave according to a model, as

\[ g_0(\omega_m) = g(x_0, \omega_m) + e, \]

(2)

where the model \( g \) depends on the set of unknown parameters denoted \( x_0 \) and \( e \) is a model of the measurement error, assumed to be Gaussian. In the present paper the observation model is complex-valued, therefore the error \( e \) is also complex-valued and its real and imaginary parts are both assumed Gaussian.

The unknown parameters \( x_0 \) are considered as the true properties of the observed system. The aim of the paper is to obtain an estimate \( \tilde{x}_0 \) of \( x_0 \) by means of model inversion.

2.2 Full frequency range parameter estimation

The model inversion procedure consists in finding the set of parameters \( \tilde{x}_0 \) that minimises the difference between the observed response and the modelled response. The 2-norm distance

\[ f_0(x, M) = \sum_{m=1}^{M} |g(x, \omega_m) - g_0(\omega_m)|^2 \]

(3)

is here chosen. The estimate \( \tilde{x}_0 \) of the unknown model parameters \( x_0 \) is then

\[ \tilde{x}_0 = \arg \min_x (f_0(x, M)). \]

(4)
Equation (4) represents a classical parameter search using the full available frequency range $\omega_m \ (m \in [1, M])$, where $x_0$ is the global minimum of Eq. (3).

A major difficulty arises in the case of resonant problems or problems exhibiting spatial or parametric symmetry. Indeed, a full-frequency parameter estimation in such cases often results in a series of local minima. These arise as a consequence of the similarity of the system’s behaviour at different values of its properties. The direct inversion of the model is therefore not viable as the optimisation procedure is not guaranteed to terminate at the global solution. It is this observation that motivates the incremental approach presented below.

2.3 Incremental-frequency parameter estimation

In order to overcome the non-uniqueness of solutions due to resonant or symmetric behaviour, an incremental method is proposed here. The method consists in guiding the parameter search towards the global solution of the problem by solving a sequence of sub-problems progressively refining the estimation. The procedure is suitable for a wide range of problems where the aim is to estimate physical properties from the knowledge of a frequency-dependent response.

The initial sub-problem is defined by constraining Eq. (3) to a narrow frequency range in which the system can be expected to behave in an asymptotic manner, that is, with a dependence on a subset of the model variables and away from resonant behaviour. The solution to such sub-problem is used as the starting point for a second sub-problem, whose frequency range is wider. The process is continued until the frequency range is that of the original full problem. The strength of such a process lies in the fact that the solution to the first sub-problem lies in a subspace that contains the true solution.

The solution of a given sub-problem can be formally stated as

$$\tilde{x}_0^{(p)} = \arg \min_x (f_0(x, M(p))), \quad p \in [1, P],$$

where $M(p)$ is the number of frequencies considered in sub-problem $p$. The sequence of sub-problems is considered such that $M(p + 1) > M(p)$, with $M_P = M$. Accordingly, the final sub-problem $P$ corresponds to the full-range problem and the final solution $x_0^{(P)} \simeq x_0$ is expected to provide a reliable estimate of the true model parameters.

Typically for dynamic problems, a low-frequency asymptotic case can be chosen as the initial sub-problem. This is the choice in the present paper and in Refs. [3, 4]. Alternatively, a high-frequency asymptotic case may be used.

3 Model inversion tools

This section describes the two inversion frameworks used in the present work, namely based on a deterministic or a statistical description. It is worth noting, nonetheless, that the scope of the present paper is intended to be independent from the optimisation tool chosen.

3.1 Gradient-based parameter search

The first inversion procedure is based on a gradient-based parameter search, formulated as a minimisation problem. The Globally Convergent Method of Moving Asymptotes (GCMMA) [5] is used here as a general-purpose optimisation algorithm, suitable for problems involving a large number of unknowns. The gradients are computed using a finite difference scheme with a differentiation step $10^{-7}$, relative to the support of $g$, i.e. the range specified for unknowns $x$. The solution is considered stable after 4 iterations where both the objective function and the design variables vary less than $10^{-4}$. In addition, no
limits are imposed upon the maximum number of inner or outer GCMMA iterations. The reader is referred
to [4, 2] for more details on the usage of GCMMA for the present type of problems.

3.2 Statistical inversion framework

In addition to the deterministic gradient-based parameter search, a Bayesian statistical inversion
framework [6, 7] is used, where all quantities of interest are modelled as random variables. The Bayesian
inference is based on the Bayes’ rule and provides the posterior probability density function of the
unknowns $x$, given an observation $g_0$. The posterior density can be used for computing point estimates,
e.g. conditional mean and maximum a posteriori (MAP), as well as estimate uncertainties.

The sampling of the posterior densities is here achieved by means of a Markov chain Monte Carlo
method, using the Metropolis-Hastings algorithm with an adaptive proposal distribution scheme [8, 9]. As
a point estimate for the parameters, we use the MAP and as an indication of the uncertainty in the point
estimates we use the narrowest 95% credible interval. For a more detailed discussion of the Bayesian
framework used in this work, we refer to [4]. The present paper extends the use of the method therein
to a complex-valued observation model (2), here implemented by concatenating real and imaginary parts
without loss of generality.

4 Application: resonant chamber coupled to a poroelastic material

4.1 Model

The method is here applied to the retrieval of the geometrical and material properties of an expansion
chamber coupled to a rigidly backed sample of poroelastic material. Figure 1 illustrates the geometry of the
problem. The observed quantity is the total impedance of the coupled system. The choice of this problem
is motivated by the challenge that it poses, in particular regarding the observability of the stiffness of the
frame of the poroelastic material. The observation and measurement noise are here numerically simulated.

![Figure 1: Expansion chamber coupled to rigidly backed poroelastic material.](image)

The expansion chamber is assumed cylindrical and modelled by means of its transfer matrix,
characterised by its length $l$ and radius $r$, both of which are assumed unknown. The tube radius is set
to $R = 40$ mm elsewhere. The transfer matrix of the expansion chamber is then

$$
T(\omega, l, r) = \begin{bmatrix}
\cos(kl) & -\frac{\rho c}{i\pi r^2} \sin(kl) \\
\frac{\rho c}{i\pi r^2} \sin(kl) & \cos(kl)
\end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}.
$$

The poroelastic material is modelled by means of its impedance, as obtained in Refs. [10, 11], to which
the reader is referred for more details. The unknown properties of the poroelastic material are the porosity
$\phi$, the flow resistivity $\sigma$ and the Young’s modulus of the solid frame $K_s$ and length $l_{PEM}$. The following
properties of the poroelastic material are considered known: tortuosity $\alpha_\infty$=1.1, viscous characteristic
length $\Lambda = 85 \, \mu m$, thermal characteristic length $\Lambda' = 170 \, \mu m$, equivalent fluid density $\rho_{eq} = 9.4 \, \text{kg} \cdot \text{m}^{-3}$ and Poisson’s ratio $\nu = 0$. The surface impedance of the rigidly backed poroelastic material sample is denoted $Z_{PEM}(\omega, \phi, \sigma, K_s, l_{PEM})$, the full derivation of which is detailed in Ref. [11].

The model can be written in terms of the surface impedance of the compound system as

$$g(x, \omega) = Z_{tot}(\omega, l, r, \phi, \sigma, K_s, l_{PEM}) = \frac{Z_{PEM}T_{11} + T_{12}\pi r^2}{Z_{PEM}T_{21}/\pi r^2 + T_{22}},$$

(7)

where the unknowns are gathered in the form

$$x = \{l, r, \phi, \sigma, K_s, l_{PEM}\}.$$  

(8)

Let $x_0$ be the set of true model parameters, whose values are arbitrarily chosen as listed in Tab. 1. Furthermore, the noise $e$ is assumed zero-mean Gaussian with a standard deviation $\varsigma = 100 \, \text{kg} \cdot \text{s}^{-1} \cdot \text{m}^{-2}$ for both real and imaginary parts.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>True value</th>
<th>Search range</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>0.4</td>
<td>[0.2, 1]</td>
<td>m</td>
</tr>
<tr>
<td>$r$</td>
<td>0.07</td>
<td>[0.04, 0.2]</td>
<td>m</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.99</td>
<td>[0.8, 1]</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>10</td>
<td>[4, 15]</td>
<td>kN\cdot s\cdot m^{-4}</td>
</tr>
<tr>
<td>$K_s$</td>
<td>400</td>
<td>[300, 500]</td>
<td>kPa</td>
</tr>
<tr>
<td>$l_{PEM}$</td>
<td>0.1</td>
<td>[0.05, 0.2]</td>
<td>m</td>
</tr>
</tbody>
</table>

Table 1: Chosen values for target model parameters and search ranges.

4.2 Full range estimation

Figure 2 depicts the solution for an ensemble of 1000 runs of the gradient-based inversion, using the full available frequency range. For each run, the initial values of the unknowns are randomly generated from a uniform distribution in the search range. It can be observed that the parameter search in a full frequency range scenario is hindered by the existence of local minima. As a matter of fact, the full-range estimation provides multiple possible values for the model parameters and therefore does not guarantee the termination of the optimiser at the true solution.

Figure 2: Normalised histograms of parameter values for 1000 full-range runs using the gradient-based search. The dotted lines represent the true solution.

4.3 Incremental estimation

Table 2 summarises the frequency ranges used for the different steps in the incremental approach. For each step, the solution is used as the starting point for the next frequency range. Figure 3 depicts the solution for an ensemble of 1000 runs of the incremental approach, using the gradient-based inversion. It can be observed that the estimation is significantly more consistent using the incremental approach. Indeed, the problem is significantly less affected by local minima in the objective function and the model parameters are guided towards the true solution by the sequence of frequency increments.
<table>
<thead>
<tr>
<th>Step</th>
<th>( f_{\text{min}} ) (Hz)</th>
<th>( f_{\text{max}} ) (Hz)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>400</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>800</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>2000</td>
</tr>
</tbody>
</table>

Table 2: Frequency ranges for the different steps of the incremental estimation

<table>
<thead>
<tr>
<th>Step</th>
<th>( l ) (cm)</th>
<th>( r ) (cm)</th>
<th>( \phi )</th>
<th>( \sigma ) (kN·s·m(^{-4}))</th>
<th>( K_s ) (kPa)</th>
<th>( l_{\text{PEM}} ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>0.8</td>
<td>4</td>
<td>300</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>0.85</td>
<td>6</td>
<td>350</td>
<td>8</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>0.9</td>
<td>8</td>
<td>400</td>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>0.95</td>
<td>10</td>
<td>450</td>
<td>12</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>1</td>
<td>12</td>
<td>500</td>
<td>14</td>
<td>100</td>
</tr>
<tr>
<td>6</td>
<td>120</td>
<td>1.05</td>
<td>14</td>
<td>600</td>
<td>16</td>
<td>120</td>
</tr>
</tbody>
</table>

Figure 3: Normalised histograms of parameter values for 1000 incremental frequency approach runs using the gradient-based search. Frequency steps 1 to 6 are shown from top to bottom. • • • True solution

### 4.4 Bayesian inversion

Figures 4 and 5 show the marginal densities at the end of steps 1 and 6, respectively. A strong correlation between the length and radius of the expansion chamber can be observed in the densities at the end of step 1. This behaviour suggests that for low frequencies the problem is only sensitive to the volume of the chamber, whose value is invariant along the marginal density in the \((l, r)\) plane. The uniformity of the marginal densities relative to the poroelastic material parameters suggests a low sensitivity of the low-frequency solution to their variation. The densities at the end of step 6 incorporate a sensitivity to all problem variables and the point estimates successfully identify all parameter values within the credible intervals.

### 5 Conclusion

An incremental approach has been presented for the estimation of the properties of coupled resonant systems. The approach consists in performing the parameter search sequentially in increasingly broad frequency ranges, where the solution for a given step is used as the starting point for the next. The approach has here been applied to the estimation of the dimensions and material properties of an expansion chamber coupled to a rigidly based sample of poroelastic material. The chosen problem poses a great challenge to a
classical full-range estimation, as numerous local minima hamper the retrieval of the true parameter values. These are due to the coupled nature of the problem and to its resonant behaviour. The proposed approach is able to guide the solution towards the true parameter values, thus avoiding local minima otherwise predominant in the estimations.

The problem is here addressed from two complementary perspectives by means of two different model inversion approaches, namely a deterministic gradient-based parameter search and a MCMC method in the Bayesian framework. While the deterministic framework proves the feasibility of the approach with a reduced number of model evaluations, the statistical inversion framework provides insight into the evolution of the sensitivity of the problem to the different model variables throughout the multiple steps.

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