ACOUSTIC METAMATERIAL DESIGN FOR LOW FREQUENCY SOUND ABSORPTION IN LINEAR AND NONLINEAR REGIMES

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An acoustic metamaterial absorber comprised of periodically distributed cavities has been built and tested with focus on low frequency sound absorption in linear and nonlinear regimes. The absorber is made up of a series of piled up pancake-like flat cavities saturated by air, separated by thin walls and traversed by a perforation at their centre. Viscous and thermal boundary layer thicknesses near the solid walls are small compared to microstructure dimensions, however the losses are non-negligible. This results in energy attenuation and abrupt wave speed reduction in the central perforation or main pore. A model for absorber effective properties is developed and compared with experimental data. The model is used to predict the frequency and the peak value of the absorption coefficient at resonance, depending on the geometrical parameters of the structure. Optimum design of the pancake structure enables to obtain the high absorption at the desired frequency. The central pore radius and the size of the cavities are shown to impact the frequency of the absorption peak. Nonlinear properties of the absorber are investigated using sine wave excitation around the resonance, which allows to obtain variations of the absorption coefficient with the amplitude of the incident wave. The flow resistivity is shown to be of great influence on the linear and nonlinear behaviour of the resonator. Flow resistivity measurements on a pancake resonator at low and higher flow rates show that the periodic set of cavities does not modify significantly the flow resistivity when compared to a simple perforated structure with same thickness.

Keywords: dead-end pores, low frequency, sound absorption

1. Introduction

In recent work [1] a microstructure was developed that contains a periodic arrangement of thin plates with central perforation separated by cavities (Figure 1), the so-called pancake absorber. The pancake absorber is one of the realizations of the microstructure design suggested earlier in [2]-[3], where the additional laterally arranged pores are coupled with the main perforation. Low value of sound speed
through the perforation achieved in these structures leads to the existence of the absorption peaks at low frequencies. In [1] a transfer matrix model (TMM) is used to predict the frequency dependence of the absorption coefficient of the absorber, which gives a good agreement with the measurements. The pancake absorbers are proven to be effective for low frequency sound absorption at only a few centimetres in thickness.

Pancake structures can be made mechanically robust and do not contain any fibrous materials or foams. This makes them good candidates for the use in hostile environments for attenuation of high amplitude noise [4]. It is known that the dominating nonlinearity in rigid porous materials is of Forchheimer’s type, associated with the growth of static flow resistivity with the Reynold’s number (and consequently pressure amplitude) [5]-[6]. If the pancakes absorbers are to be used for attenuation of the high amplitude sound, it is essential to investigate their acoustical properties in the nonlinear regime. TMM model in its standard form is not suitable for dealing with the nonlinearity, so a new model needs to be developed.

The aim of this work is to investigate the sound absorption properties of the pancake structures in the nonlinear regime both theoretically and experimentally. We start with developing a linear model for the effective properties of the absorber and use it to predict the dependence of the positions of the lowest resonance frequencies and the peak values of the absorption coefficient. This model is validated against the experimental results obtained in standard impedance tube using white noise excitation. Once validated, the model is generalised to include Forchheimer’s nonlinearity. The measured dependence of flow resistivity on the flow rate is used in this model. The nonlinear model predicts how the peak value of the absorption coefficient varies with the amplitude of the incident wave. It is validated by performing measurements using sine waves in specially designed high sound pressure level (HSPL) impedance tube. The highest-pressure level of the incident wave achieved in the tube is around 160 dB.

![Figure 1: Pancake absorber comprised of an inner periodic arrangement [1]](image)

2. Model

2.1 Linear case – low sound amplitude

The structure with the lateral annual cavities is treated as double porosity material of high permeability contrast [7]. In this case, the normalised effective density of the double porosity material \( \rho_{db} \) remains equal to that of the mesodomain (i.e. the central pore with radius \( r_0 \)):

\[
\rho_{db} = \frac{\rho_p}{\phi_p}
\]  

(1)

The validity of this assumption will be confirmed experimentally. Porosity of the mesodomain is defined as the ratio of the main pore surface area to that of the absorber’s face with radius \( R \), \( \phi_p = \frac{r_0^2}{R^2} \). \( \rho_p \) is normalised effective density of fluid in the central pore.

Due to the presence of the lateral cavities (microdomain), the normalised effective bulk modulus of the structure \( K_{db} \) is calculated as \( K_{db}^{-1} = \phi_p K_p^{-1} + (1 - \phi_p) \phi_m F_d(\omega) K_m^{-1} \). Here \( K_p \) and \( K_m \) are effec-
tive bulk moduli of fluid in meso and microdomains normalised by \( \gamma P_0 \), \( \phi_m \) is porosity of the microdomain, \( P_0 \) is atmospheric pressure and \( \gamma \) is adiabatic constant. \( F_d(\omega) \) is the pressure diffusion function equal to the pressure averaged over the microdomain divided by that in mesodomain. It is calculated using equation (8) of \cite{8}, \( F_d(\omega) = 1 + k_m^2 \frac{D(\omega)}{1-\phi_p} \), where the requirement of sound propagation in microdomain being isothermal and in viscous regime is relaxed. In the case of the considered absorber geometry \( D(\omega) = \frac{2}{R^2} \int_0^R \bar{D} \, r \, dr \), where \( \bar{D} \) is calculated from equation \( \partial_r \bar{D} + \frac{1}{r} \partial_r \bar{D} + k_m^2 \bar{D} = -1 \) with boundary conditions \( \bar{D}(r_0) = 0 \) and \( \partial_r \bar{D}(R) = 0 \). Here \( k_m \) is wavenumber in the microdomain and \( r \) is the polar coordinate. The first boundary condition simply follows from continuity of pressure at the boundary of meso and microdomains \( r = r_0 \) while the second condition is at the rigid boundary \( r = R \). Solution of this boundary value problem is \( \bar{D} = \frac{1}{k_m^2} \left( \frac{J_0(k_m r_0)}{J_0(k_m R_0) - SH_0(k_m r_0)} - 1 \right) \), which, after averaging, results in the following pressure diffusion function \( F_d(\omega) = -\frac{2\phi_p}{1-\phi_p} \frac{J_1(k_m r_0)}{k_m r_0 J_0(k_m r_0) - SH_0(k_m r_0)} \), where \( S = \frac{J_2(k_m R)}{H_m(k_m R)} \). It is convenient to introduce the surface admittance of the wall (microdomain) normalised by admittance of air \( G = -\frac{\rho_0 c}{i z_m} \frac{J_1(k_m r_0) - SH_0(k_m r_0)}{J_0(k_m r_0) - SH_0(k_m r_0)} \), where \( z_m \) is characteristic impedance of the microdomain, time convention \( e^{-i \omega t} \) is used. Then, the pressure diffusion function can be expressed through \( G \), which leads to the following final equation for the effective bulk modulus of the structure:

\[
\frac{1}{K_{db}} = \phi_p \left( \frac{1}{K_p} + \frac{2iG}{kr_0} \right)
\]

In equations (1) and (2), \( K_p \) and \( \rho_p \) are calculated using standard Johnson – Champoux-Allard model \cite{9} for the cylindrical pore and \( k_m \) and \( z_m \) are calculated using the same model for the arrangements of slits with porosity \( \phi_m = \frac{2d}{2d+h} \) with \( 2d \) being the spacing between the plates and \( h \) being the plate thickness.

To estimate the frequency of the first resonance \( f_r \) of the hard-backed layer with thickness \( L \) the following assumptions are used. First it is assumed that \( k_m R \ll 1 \) and leading order expansions are used for the Bessel functions. Second, resonance frequency is assumed to be much higher than characteristic viscous and thermal frequencies \( \omega_{b(m,p)} \) and \( \omega_{b(m,p)}' \). For the circular pore and slits, this means that \( f_r \gg \frac{1}{2\pi d^2 \rho_0} \) and \( f_r \gg \frac{1}{2\pi} \frac{16\eta}{\rho_0 r_0^5} \), here \( \eta \) and \( \rho_0 \) are viscosity coefficient and density of air. For the dimensions of the samples, considered in this paper, this assumption is valid for frequencies higher than 10 Hz. Third, it is required that \( \phi_m \gg \phi_p \), which is justified by the use of central perforation much smaller than the cross section of the sample. The estimation results in the following expression for the first resonance frequency

\[
f_r \approx f_0 \sqrt{\frac{\phi_p}{\phi_m}} \left( 1 - \frac{1}{2} \sqrt{\frac{f_{\delta v}}{f_0}} \frac{\phi_m}{\phi_p} \left( 1 + \frac{\gamma - 1}{\sqrt{\gamma \rho_0 r_0}}, \frac{r}{2d} \right) \right)
\]

where \( f_0 = \frac{c}{4L} \) is the frequency of the quarter wavelength resonance of air slab of thickness \( L \) and \( f_{\delta v} = \frac{\eta}{\rho_0 r_0^5} \) is the frequency at which the viscous boundary layer thickness is equal to the pore radius \( r_0 \).

Under the same assumptions absorption coefficient at resonance is estimated as
\[ \alpha_r \approx 4 \frac{1 - e^{-2n}}{\left( \sqrt{\phi_m \phi_p} + 1 \right) - e^{-n} \left( 1 - \sqrt{\phi_m \phi_p} \right)^2} \]  

(4)

where \( n \approx \frac{\pi}{2} \left( 1 + \frac{y^{-1} r}{\sqrt{\phi_p r}} \right) \). Frequency dependence of the absorption coefficient of the structure is calculated as \( \alpha(\omega) = 1 - |R|^2 \), where \( R = \frac{1 - z_s}{1 + z_s} \) and \( z_s = i\sqrt{\rho_{db} K_{db}} \cot \left( kL \sqrt{\rho_{db} K_{db}} \right) \). This completes the linear model for the acoustic properties of the pancake absorber.

### 2.2 Nonlinear case – high sound amplitude

For simplicity a scaling function introduced in [6] is used for \( \rho_{db} \). Accounting for the Forchheimer’s nonlinearity, the equation (1) is generalised as

\[ \rho_{nl} = \rho_{db} + \frac{\sigma_0}{-i\omega \rho_0 \phi_p} \xi |v| \]  

(5)

where a linear dependence of flow resistivity on the particle velocity amplitude is assumed \( \sigma(v) = \sigma_0 (1 + \xi |v|) \), where Forchheimer’s nonlinearity parameter \( \xi \) is measured in flow resistivity test and \( \sigma_0 = \frac{\eta}{r_0^2} \) is flow resistivity of the central pore. Normalised surface admittance of the structure \( G = \frac{1}{z_s} \) dependence on the incident pressure amplitude \( P_i \) can be found from the following equations:

\[ G' - iq \rho_{db} G^2 + i q K_{db}^{-1} a |v| G^2 = 0 \]  

(6)

\[ \frac{v'}{v} = -iq \frac{K_{db}^{-1}}{G} \]  

(7)

where \( v(y) \) is particle velocity, \( a = \frac{\sigma_0 L \xi}{\rho_{oc}} \), \( q = \frac{\omega L}{c} \) and the differentiation is performed with respect to the normalised spatial variable \( y = 1 - \frac{z}{L} \), so that \( y = 0 \) corresponds to the backing of the sample and \( y = 1 \) to the front face. Equations (6)-(7) are considered with boundary conditions \( G(0) = 0 \) and \( v(1) = \frac{P_i}{\rho_{oc} \left( 1 + G(1) \right)} \) \( (P_i \) is the incident pressure amplitude in Pa) and are solved by iterations. First, a linear case \( \xi = 0 \) is considered and \( G(1) \) is found from equation (6). This is then used in the boundary condition for \( v(1) \) and a new value of \( G(1) \) is found. The process is repeated until a good convergence is achieved.

### 3. Experiments and comparisons with models

All measurements reported here have been performed on a pancake absorber obtained by stacking perforated plates and comprising 3 mm plates, separated by 3 mm cavities. The outer radius of the absorber is 5 cm and the radius of the central pore is 4 mm.

#### 3.1 Flow resistivity and Forchheimer’s parameter measurements

Measurements of the flow resistivity have been performed with two aims. The first is to justify the use of the high permeability contrast assumption (1) and the second is to measure Forchheimer’s nonlinearity parameter \( \xi \) required by the nonlinear model.
For the first aim, airflow resistivity of the pancake absorber was measured at low flow rate. After that the absorber was disassembled and the measurements have been repeated for the stack of perforated plates (without lateral cavities). The results are shown in Figure 2 (a-b). The measured flow resistivity value for the pancake absorber is $\sigma_0 = 1484 \frac{Pa \cdot s}{m^2}$, while for the stacked plates it is $1479 \frac{Pa \cdot s}{m^2}$. The theoretical value for the straight cylindrical pore with $r_0 = 4\, mm$ is $1405 \frac{Pa \cdot s}{m^2}$. The error for stacked plates and pancake structure is 5.3% and 5.6% respectively. The presence of cavities ($2d = 3\, mm, h = 3\, mm$) appears to have little or no effect on static flow resistivity which confirms high permeability contrast assumption adopted in the model. The results of the measurements where both stacked plates and pancake absorber are exposed to much higher flow rates are shown in Figure 3, c. Forchheimer’s nonlinearity parameter can be determined from a linear approximation of flow resistivity dependence on the flow rate. The presence of cavities appears to have little or no effect for flow resistivity of samples even for higher flow rates. The determined Forchheimer’s parameter value is $\xi = 258.6 \frac{s}{m}$.

![Graphs](image.jpg)

Figure 2. a) Flow resistivity of the pancake absorber at low flow rates b) Flow resistivity of the stacked plates at low flow rates and c) Flow resistivity of pancake absorber (black) and stacked plates (red) at high flow rates.
3.2 Impedance tube measurements – low amplitude sound

Absorption coefficient of the rigidly terminated pancake structure has been measured in a standard impedance tube, using two microphone method. The frequency range was 100 Hz-1600Hz.

![Graphs showing absorption coefficient data and model predictions for low amplitude sound measurements.](image)

Figure 3. Absorption coefficient data (markers) and model predictions (grey lines) for a) 6 cm absorber and b) 3 cm absorber. Broken line shows model predictions accounting for the end correction. Red line shows absorption coefficient predictions for stacked plates, thickness 6 cm.

Comparisons between the model described in Section 2.1 and measurements for two thicknesses of the sample, 6 cm and 3 cm, are shown in Figure 3 (a,b). Good agreement between the measurements and the model are observed for both samples in the low frequency range. For 6 cm sample, equations (3) and (4) give first resonance frequency $f_r = 156.2$ Hz and peak absorption coefficient value $\alpha_r = 1.00$. The measured values are $f_r = 150.0$ Hz and $\alpha_r = 0.96$.

It is clear, that the presence of cavities shift the position of the first peak dramatically. The same thickness slab with a single perforation would have absorption peak at 1423 Hz – this is 9.4 times higher than the resonance frequency of the absorber.

For 3cm sample, equations (3)-(4) predict $f_r = 315.9$ Hz and $\alpha_r = 0.97$. However, the measured values are $f_r = 295.0$ Hz and $\alpha_r = 0.92$. However, the influence of the end correction is significant considering the small thickness of the sample and a relatively large size of the central pore. Model results accounting for the end correction are shown in grey broken lines in Figure 3, b. In this case, the model predicts $f_r = 284.5$ Hz and $\alpha_r = 0.98$, which is closer to the measured values.

The main explanation for discrepancies between theory and experiments at higher frequencies is the effect of structural resonance of the periodic structure (not accounted for in the model), also observed in [1], which creates a counter effect or damping on the acoustic resonances responsible for absorption peaks. This also explains why, starting from the first resonance at lowest frequency, the match with theory for the absorption amplitudes gradually diminishes as the vibration effect is itself gradual. This effect is in the range 700 - 1300 Hz for 6cm sample and 600 - 1300 Hz for the 3cm thickness.

3.3 Impedance tube measurements – high amplitude sound

The measurements for high amplitude sound have been performed using a specially designed impedance tube at the ISAT, University of Burgundy, France. Sine wave excitation has been used for a better control over the amplitude of the incident wave. For the usual white noise excitation, there is no guarantee
that each frequency component has the same amplitude. The results of the measurements for several frequencies around the resonance performed in the incident pressure range 0.6-230 Pa are compared with the model described in Section 2.2. Forchheimer’s parameter value measured during the flow resistivity tests has been used in the model. Results for three frequencies, 146 Hz, 151 Hz and 156 Hz, around the resonance of 6 cm sample are shown in Figure 4 (a-c) and compared with the model predictions. Ten iterations have been used in the solution. It is clear, that the dependence of the peak absorption coefficient value on the incident pressure amplitude is strong and progressively increases with amplitude strength. It drops by 46% (from 0.96 to 0.52) as the incident wave amplitude increases to 236 Pa (SPL=138.4dB). The agreement between the model and the measurements is satisfactory for the sine wave set up.

**Figure 4. Absorption coefficient dependence on incident pressure amplitude for 6 cm absorber, markers – data, lines – model given by equations (6)-(7) predictions. a– 146 Hz, b – 151 Hz (first resonance), c – 156 Hz, d – predictions of absorption coefficient versus frequency dependence for $P_i = 236 \, Pa$ (black line) compared with linear model (grey lines)**

To illustrate how the increase of the incident wave amplitude affects the overall shape of the absorption curve, equations (6)-(7) have been solved for different frequencies assuming that each frequency component has the same amplitude for $P_i = 0.01 \, Pa$ (linear case) and $P_i = 236 \, Pa$. The results are
shown in Figure 4,d. The effect of nonlinearity does not change the frequency of the absorption peaks, while strongly decreasing the peaks.

4. Conclusion

An absorbing structure comprised of periodically arranged metallic plates with the central perforation separated with air cavities, similar to that described in [1], has been built and tested in both linear and nonlinear regimes. The results of measurements are compared with the models – the analytical linear model and a numerical model accounting for Forchheimer’s type nonlinearity. A good agreement is demonstrated between the model and the absorption coefficient data in the linear regime, i.e. for low levels of the incident sound. Simple expressions (4)-(5) adequately describe the frequency of the first peak and the peak absorption coefficient value. It is shown that the absorption peak frequencies weakly depend on the incident pressure amplitude, while the peak values are strongly attenuated as pressure grows. This work confirms, that Forchheimer’s nonlinearity should be accounted for if structure is to be used for attenuating high amplitude sound and provides the model for doing this.

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