ACOUSTIC ABSORPTION IN A SUBSONIC MEAN FLOW AT A SUDDEN CROSS SECTION AREA CHANGE

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Acoustic waves propagating within a duct containing a subsonic mean flow are of interest in many practical applications such as aeroengines or gas turbines. In these systems, sudden cross section area increases where flow separation occurs are widespread and the acoustic energy balance is known to be affected for such conditions. On the other hand, sudden cross section area decreases are usually assumed to be isentropic and the acoustic energy balance is unaffected. The objective of this work is to determine the acoustic absorption coefficient for various models of sudden cross section area increase and decrease commonly implemented in low order network tools used for thermoacoustic stability prediction. Analytical expressions in the low Mach number limit are also provided and compared with numerical predictions. It is shown that for a sudden cross section area increase with flow separation, the acoustic absorption coefficient depends on the upstream Mach number, cross section area ratio and boundary acoustic reflection coefficient only. For certain values of these parameters, all of the acoustic energy is damped across the area change. On the other hand, the acoustic energy is amplified across the area change for other values of these parameters.

Keywords: acoustic absorption, sudden area change

1. Introduction

In many engineering applications, such as aeroengines or gas turbines [1, 2], the cross sectional area of a duct sustaining a subsonic mean flow may suddenly change, which may affect the acoustic energy balance of the system [3, 4, 5]. Moreover, it is known that unstable thermoacoustic modes will develop inside combustors when the acoustic sources are not exactly compensated by acoustic damping [6, 7]. As a consequence, predicting the absorption or amplification of acoustic waves at sudden area changes is critical for the prediction of the thermoacoustic stability of combustors. The main objective of this work is to quantify the damping or amplification of acoustic waves [8, 9] across sudden cross section area changes based on the models commonly implemented in low order network tools [10].

The expressions of the acoustic absorption coefficient in terms of the coefficients of the S-matrix, also known as the Scattering Matrix, are first derived in Sec. 2. The absorption coefficient generated by a sudden area change filled with a quiescent fluid is then investigated in Sec. 3. Then, the quiescent fluid is replaced by an isentropic flow in Sec. 4. Finally, the absorption coefficient corresponding to an area increase sustaining a subsonic mean flow with flow separation is examined in Sec. 5.
2. Acoustic absorption coefficient in terms of the S-Matrix coefficients

The scope of this work is limited to the absorption of acoustic waves at sudden cross section area changes sustaining a subsonic mean flow. The fluid is assumed to be a perfect gas. The retained geometry might be either an area increase or an area decrease, as depicted in Fig. 1-(Left) and Fig. 1-(Right) respectively. The mean flow direction is from left to right, and the sudden cross section area change is located at $x = 0$. It is assumed that between the upstream and downstream locations, denoted by $x = u$ and $x = d$ respectively, the system is acoustically compact. Moreover, the mean and acoustic variables are assumed to be one-dimensional at $x = u$ and $x = d$. All the variables defined at $x = u$ and $x = d$ will be denoted by indices $u$ and $d$ respectively. The overbar ($\bar{\cdot}$) and prime (') denote mean and acoustic variables respectively. In addition to the cross section area $A$, the mean flow velocity $\bar{u}$, mean speed of sound $\bar{c}$, mean temperature $\bar{T}$, mean pressure $\bar{p}$ and mean density $\rho$ may be modified across the area change. The expressions presented in this work are valid for both harmonic conventions $\exp(\pm i\omega t)$. The Riemann invariant associated with the positive (respectively negative) direction is denoted by $f$ (respectively $g$). Based on dimensional analysis, it is shown that the mean flow is fully described by five dimensionless parameters defined as:

\[
M_u = \frac{\bar{u}_u}{\bar{c}_u} \quad || \quad M_d = \frac{\bar{u}_d}{\bar{c}_d} \quad || \quad \theta = \frac{A_u}{A_d} \quad || \quad \alpha = \frac{\bar{p}_u}{\bar{p}_d} \quad || \quad \Xi = \sqrt{\frac{\bar{T}_u}{\bar{T}_d}} \quad (1)
\]

Where $M_u$ (respectively $M_d$) is the Mach number associated with the mean flow assessed at $x = u$ (respectively $x = d$), $\theta$ represents the cross section area ratio, and $\alpha$ and $\Xi$ quantify the mean pressure and mean temperature ratio between $x = u$ and $x = d$.

The acoustic absorption coefficient $\Delta$ is then defined as [8, 9]:

\[
\Delta = 1 - \frac{|W_{f_d}| + |W_{g_u}|}{|W_{f_u}| + |W_{g_d}|} \quad (2)
\]

Where $W_i$ is the acoustic energy flux associated with the Riemann invariant $i$ [8, 9]. Since the acoustic waves at the upstream $x = u$ and downstream $x = d$ locations are assumed to be plane waves, the acoustic pressure $p'$ and acoustic velocity $u'$ can be expressed in terms of $f_u$, $g_u$, $f_d$ and $g_d$ at these locations. The
acoustic energy flux in the presence of a mean flow is then computed at the upstream and downstream locations and it is finally shown that:

\[
\Delta = 1 - \frac{(1 + M_d)^2 |f_d|^2 + \theta \Xi \alpha^{-1} (1 - M_u)^2 |g_u|^2}{\theta \Xi \alpha^{-1} (1 + M_u)^2 |f_u|^2 + (1 - M_d)^2 |g_d|^2}
\] (3)

Based on the definition of the acoustic absorption coefficient \(\Delta\), several possibilities arise:

- \(\Delta = 1\) - All of the acoustic energy going into the system is damped.
- \(0 < \Delta < 1\) - Part of the acoustic energy entering the system is damped. An increase in \(\Delta\) corresponds to more acoustic damping.
- \(\Delta = 0\) - The acoustic energy balance is unaffected.
- \(\Delta < 0\) - The acoustic energy entering the system is amplified. A decrease in \(\Delta\) corresponds to more acoustic amplification.

The S-Matrix, denoted by \(S\) and sometimes called the Scattering Matrix in the literature [11], is then used to relate the Riemann invariants upstream and downstream the cross section area ratio:

\[
\begin{pmatrix}
  f_d \\
g_u
\end{pmatrix} =
\begin{pmatrix}
  S(1,1) & S(1,2) \\
  S(2,1) & S(2,2)
\end{pmatrix}
\begin{pmatrix}
  f_u \\
g_d
\end{pmatrix}
\] (4)

An additional equation is necessary in order to express the acoustic absorption coefficient \(\Delta\) as a function of the S-Matrix coefficients. If the acoustic reflection coefficient \(R\) is known at \(x = u\) (respectively \(x = d\)), then this equation is given by \(f_u = R_u g_u\) (respectively \(g_d = R_d f_d\)). Using these equations along with Eqs. (3)-(4), the expressions of the acoustic absorption coefficient in terms of the upstream \(R_u\) or downstream \(R_d\) acoustic reflection coefficient are given by:

\[
\Delta_{R_u} = 1 - \frac{(1 + M_d)^2 |S(1,2)|^2 + R_u \det S|^2 + \theta \Xi \alpha^{-1} (1 - M_u)^2 |S(2,2)|^2}{\theta \Xi \alpha^{-1} (1 + M_u)^2 |R_u S(2,2)|^2 + (1 - M_d)^2 |1 - R_u S(2,1)|^2}
\] (5)

\[
\Delta_{R_d} = 1 - \frac{(1 + M_d)^2 |S(1,1)|^2 + \theta \Xi \alpha^{-1} (1 - M_u)^2 |S(2,1)|^2 + R_d \det S|^2}{\theta \Xi \alpha^{-1} (1 + M_u)^2 |1 - R_d S(1,2)|^2 + (1 - M_d)^2 |R_d S(1,1)|^2}
\] (6)

Equation (5) (respectively Eq. (6)) should be employed when the acoustic reflection coefficient is known at \(x = u\) (respectively \(x = d\)). These two equations can be obtained from each other by using:

\[
R_d = \frac{1 - R_u S(2,1)}{S(1,2) + R_u \det S} \quad || \quad R_u = \frac{1 - R_d S(1,2)}{S(2,1) + R_d \det S}
\] (7)

### 3. Absorption coefficient of a sudden area change without mean flow

The scope of this work is limited to linear sound waves. The physical variables are expressed as the sum of a mean flow variable, denoted by an overbar, and a small-amplitude acoustic perturbation, denoted by a prime [12]. For instance, the pressure can be expressed as \(p = \bar{p} + p'\) with \(p'/\bar{p} \ll 1\). Jump conditions for the mean and acoustic variables are then obtained by considering the zeroth and first order terms of the conservation equations [1][2].
In the absence of a mean flow, corresponding to \( M_u = M_d = 0 \), the mean conservation of momentum and mean conservation of energy reduce to \( \alpha = 1 \) and \( \Xi = 1 \) respectively. All the mean flow variables remain constant across the sudden area change and the only unconstrained dimensionless parameter is \( \theta \). To the first order, the conservation of mass and momentum yield [1, 2]:

\[
\begin{align*}
    u'_d A_d &= u'_u A_u \\
    p'_d &= p'_u
\end{align*}
\]

(8)

(9)

The corresponding S-matrix is expressed in terms of \( \theta \). Using Eqs. (5)-(6), it is then shown that whatever the upstream \( R_u \) or downstream \( R_d \) acoustic reflection coefficient, the acoustic absorption coefficient is always equal to zero across a sudden cross section area change without mean flow:

\[
\Delta R_u = \Delta R_d = 0
\]

(10)

Since all dissipation processes are neglected in Eqs. (8)-(9), the ingoing and outgoing acoustic energy fluxes have to be equal and the acoustic absorption coefficient is inevitably equal to zero.

4. Absorption coefficient of a sudden area change with an isentropic flow

A subsonic isentropic mean flow is now considered between the inlet \((x = u)\) and outlet \((x = d)\) of the system. The mass flow rate, entropy and stagnation enthalpy are conserved between \(x = u\) and \(x = d\).

This model accurately represents sudden and compact area decreases sustaining a subsonic mean flow since viscous effects can be neglected in that case [1, 10]. This model is also commonly implemented in low order network tools used for thermoacoustic stability prediction to represent compact area decreases [10]. On the other hand, viscous effects cannot be neglected for sudden cross section area increases because of flow separation occurring at the rim [1, 10]. For an acoustically compact, slowly-varying area increase where flow separation does not occur, the isentropic model may still be used [13] and the results presented in this section still apply.

The conservation of mass, entropy and stagnation enthalpy lead to the expressions of \( \alpha, \Xi \) and \( \theta \) as functions of the upstream \( M_u \) and downstream \( M_d \) Mach numbers of the mean flow:

\[
\begin{align*}
    \alpha &= \left( \frac{1 + \gamma^{-1} M_d^2}{1 + \gamma^{-1} M_u^2} \right)^{\frac{\gamma - 1}{2}} \\
    \Xi &= \left( \frac{1 + \gamma^{-1} M_d^2}{1 + \gamma^{-1} M_u^2} \right)^{\frac{1}{2}} \\
    \theta &= \frac{M_d}{M_u} \left( 1 + \frac{\gamma^{-1} M_d^2}{M_u^2} \right)^{-\frac{(\gamma + 1)}{2(\gamma - 1)}}
\end{align*}
\]

(11)

The first-order terms in the conservation of mass, entropy and stagnation enthalpy then reduce to:

\[
\begin{align*}
    u'_d + M_d \frac{p'_d}{p_d \bar{c}_d} - M_d \frac{s'_d \bar{c}_d}{c_p} &= M_d M_u^{-1} \Xi^{-1} u'_u + M_d \Xi^{-1} \frac{p'_u}{\bar{p}_u \bar{c}_u} - M_d \Xi^{-1} \frac{s'_u \bar{c}_u}{c_p} \\
    (\gamma - 1) M_d u'_d + (\gamma - 1) \frac{p'_d}{p_d \bar{c}_d} + \frac{s'_d \bar{c}_d}{c_p} &= M_u \Xi (\gamma - 1) u'_u + \Xi (\gamma - 1) \frac{p'_u}{\bar{p}_u \bar{c}_u} + \Xi \frac{s'_u \bar{c}_u}{c_p}
\end{align*}
\]

(12)

(13)

(14)

Assuming that there are no upstream entropy waves \( s'_u \) entering the system, Eq. (13) shows that there are no entropy waves generated inside the system \( s'_d = 0 \). Equations (12)-(14) are then used to derive the
S-matrix. Based on Eqs. (5)-(6), the acoustic absorption coefficient is subsequently derived. Once again, it is shown that whatever the upstream $R_u$ or downstream $R_d$ acoustic reflection coefficient, the acoustic absorption coefficient of a sudden area change sustaining an isentropic mean flow is always equal to zero:

$$\Delta R_u = \Delta R_d = 0$$

(15)

This result is consistent with the isentropic flow assumption, which implies that all dissipation processes are neglected.

5. Absorption coefficient of a sudden area increase with flow separation

A subsonic mean flow experiencing flow separation at a sudden cross section area increase is not isentropic because viscous dissipation occurs in the shear layers and recirculation zones. As a consequence, the conservation of entropy used in Sec. 4 is replaced by the conservation of momentum. Stagnation enthalpy is still conserved between $x = u$ and $x = d$ because viscous effects are usually assumed to be negligible at the inlet and outlet of the system. In the end, the conservation of mass, momentum and stagnation enthalpy for the mean flow are applied, with the conservation of momentum assuming that the pressure on the lateral surface just after the sudden cross section area increase is equal to the upstream pressure $p_u$ [1]. These conservation equations become:

$$M_d \Xi = \alpha \theta M_u$$

(16)

$$1 + \gamma M_d^2 = \alpha (1 + \theta \gamma M_u^2)$$

(17)

$$1 - \Xi^2 = \frac{\gamma - 1}{2} (M_u^2 \Xi^2 - M_d^2)$$

(18)

Additionally, the mean entropy growth across the sudden cross section area increase is given by:

$$\frac{\Delta s}{c_v} = \log \left( \alpha^{\gamma - 1} \Xi^{-2\gamma} \right)$$

(19)

Equations (16)-(18) are solved symbolically, leading to expressions of $M_d$, $\Xi$ and $\alpha$ in terms of $M_u$ and $\theta$. The first-order terms of the conservation of mass, momentum and stagnation enthalpy are given by:

$$u_d' + M_d \frac{p_d'}{\rho_d c_d} - M_d \frac{s_d' c_d}{c_p} = M_d M_u^{-1} \Xi^{-1} u_u' + M_d \Xi^{-1} \frac{p_u'}{\rho_u c_u} - M_d \Xi^{-1} s_u' c_u$$

(20)

$$2M_d u_d' + (1 + M_d^2) \frac{p_d'}{\rho_d c_d} - M_d \frac{s_d' c_d}{c_p} = 2M_d u_u' + \alpha \Xi^{-1} (1 + \theta M_u^2) \frac{p_u'}{\rho_u c_u} - M_d M_u \frac{s_u' c_u}{c_p}$$

(21)

$$(\gamma - 1) M_d u_d' + (\gamma - 1) \frac{p_d'}{\rho_d c_d} + \frac{s_d' c_d}{c_p} = M_u \Xi (\gamma - 1) u_u' + \Xi (\gamma - 1) \frac{p_u'}{\rho_u c_u} + \Xi \frac{s_u' c_u}{c_p}$$

(22)

Assuming again that there are no upstream entropy fluctuations $s_u'$ entering the system, but accounting for the generation of entropy waves within the system, the expression of the S-Matrix is obtained. The expressions of the acoustic absorption coefficient of a sudden cross section area increase sustaining a subsonic mean flow with flow separation are then obtained using Eqs. (5)-(6).

Two important limiting cases are investigated, corresponding to upstream (denoted by LC1) and downstream (denoted by LC2) anechoic boundary conditions with an upstream Mach number $M_u \ll 1$. 

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Figure 2: Acoustic absorption coefficient of a sudden area increase with flow separation as a function of: (Top) the upstream Mach number for three different cross section area ratios and (Bottom) the cross section area ratio for three different upstream Mach numbers. (Left) Upstream anechoic boundary conditions. (Right) Downstream anechoic boundary conditions. The solid line represents the numerical solution while the dotted line represents the analytical solution for $\mathbf{M}_u \ll 1$.

By neglecting all Mach number terms of order larger than one in Eqs. (16)-(22), the absorption coefficient of a sudden cross section area increase with flow separation is obtained for these two limiting cases:

$$\Delta^{LC1}_{R_u} = \frac{4\theta(\theta - 1)^2}{(\theta + 1)^2}M_u$$

$$\Delta^{LC2}_{R_d} = \frac{4(\theta - 1)^2}{(\theta + 1)^2}M_u$$

The predictions of the acoustic absorption coefficient of a sudden cross section area increase with flow separation are represented in Fig. 2 for upstream (Left) and downstream (Right) anechoic boundary conditions. The dashed lines correspond to the expressions in the low Mach number limit, given by Eq. (23), while the solid lines correspond to the general solution. The absorption coefficient is either represented as a function of the upstream Mach number for three prescribed cross section area ratios (Top) or as a function of the cross section area ratio for three prescribed upstream Mach numbers (Bottom).

Figure 2-(Top) shows that the expressions of the acoustic absorption coefficient in the low Mach number limit given by Eq. (23) are in good agreement with the general solution whenever $M_u < 0.2$, except for downstream anechoic boundary conditions with a large cross section area jump. This is corroborated by Fig. 2-(Bottom), where the agreement between the general solution and the low Mach approximation is almost perfect for an upstream Mach number $M_u = 0.1$ and for upstream anechoic boundary conditions, but not for downstream anechoic boundary conditions when $\theta \to 0$. This discrepancy becomes negligible for $M_u < 0.05$. Hence, Eq. (23) can only be employed for very limited upstream Mach number flows.
Figure 3: Acoustic absorption coefficient of a sudden area increase with flow separation as a function of the phase (x-axis) and modulus (y-axis) of the upstream acoustic reflection coefficient $R_u$. (Left) $M_u = 0.1$ - (Center) $M_u = 0.5$ - (Right) $M_u = 0.9$. (Top) $\theta = 0.1$ - (Middle) $\theta = 0.5$ - (Bottom) $\theta = 0.9$.

It is also shown in Fig. 2 that for upstream or downstream anechoic boundary conditions, the acoustic energy is always damped at a sudden cross section area increase with flow separation. For downstream anechoic boundary conditions, represented in Fig. 2-(Right), the absorption coefficient increases when the upstream Mach number increases. For a given upstream Mach number $M_u$, the absorption coefficient decreases when the upstream and downstream cross section areas become closer. For upstream anechoic boundary conditions, represented in Fig. 2-(Left), the absorption coefficient features a maximum, which depends on both the upstream Mach number $M_u$ and the cross section area ratio $\theta$.

The impact of the upstream acoustic reflection coefficient $R_u$ on the acoustic absorption coefficient is finally investigated in Fig. 3 for various upstream Mach numbers $M_u$ and cross section area ratios $\theta$. It is clear that the absorption coefficient remains close to zero and relatively independent of the upstream reflection coefficient for small values of $M_u$. When $M_u$ increases, the acoustic absorption coefficient becomes highly dependent on the upstream reflection coefficient, especially when $\theta$ departs from unity. For $M_u = 0.9$ and $\theta = 0.1$, corresponding to Fig. 3-(Top Right), the acoustic absorption coefficient is almost equal to 1 for $R_u \approx -0.5$, while it becomes slightly negative for $R_u \approx 1$, thus suggesting that the acoustic energy is amplified for such conditions. For $M_u = 1$ and an arbitrary cross section area ratio $\theta$, it can be shown that the absorption coefficient is exactly equal to one for a specific value of the upstream
reflection coefficient. In that case, all of the incoming acoustic energy is damped across the sudden area increase. On the other hand, the absorption coefficient becomes slightly negative over a wide range of parameters when $M_u = 1$.

6. Conclusion

The acoustic energy balance was investigated for three sudden cross section area change models by deriving the corresponding expressions of the acoustic absorption coefficient. In the absence of a mean flow, the upstream and downstream acoustic energy fluxes were shown to be equal whatever the acoustic boundary conditions. This conclusion was extended to sudden cross section area changes sustaining a subsonic isentropic mean flow, as typically describes a sudden area decrease containing a subsonic mean flow. However, the isentropic assumption no longer holds if large scale flow separation occurs, as for a sudden area increase where the subsequent viscous dissipation in the shear layers and recirculation zones acts as an entropy source.

The acoustic absorption coefficient of a sudden cross section area increase with flow separation was investigated for upstream and downstream anechoic boundary conditions. It was shown to depend on the upstream Mach number $M_u$ and cross section area ratio $\theta$ only. Moreover, the acoustic energy was shown to be damped across the area increase whatever $M_u$ and $\theta$. For non-anechoic boundary conditions, the absorption coefficient was shown to be a function of $\theta$, $M_u$ and the upstream $R_u$ or downstream $R_d$ acoustic reflection coefficients. It was shown that for certain values of these parameters, the acoustic absorption coefficient was equal to one, indicating that all the acoustic energy was damped across the area increase. Conversely, it was demonstrated that the absorption coefficient could become slightly negative for other parameters, corresponding to an amplification of the acoustic energy across the area increase.

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