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1. Introduction

Metamaterials are a new class of artificial composites engineered to have transcendental properties that cannot found from natural materials. In the past decades, metamaterials have attracted much attention in many research fields. Metamaterials are originally introduced to tailor the electromagnetic optical waves [1-4]. Nowadays, the concept of metamaterial has expanded to the areas of acoustic/elastic metamaterials. Unusual properties such as effective negative mass and dynamic stiffness [5] and negative bulk modulus [6] can be seen from elastic/acoustic metamaterials.

An important feature of elastic/acoustic metamaterials is the existence of bandgaps within which no waves can propagate. These bandgaps are mainly caused by two phenomena, Bragg scattering and local resonance. Bragg scattering happens when the wavelength of propagating wave is of the same magnitude as the unit cell dimension. As a contrast, locally resonant bandgap relies on the resonance of internal oscillators. The local resonance bandgap is generally at frequencies lower than that of Bragg scattering bandgap. Henceforth, more metamaterial designs have been proposed with various local resonators, such as inclusions coated with soft rubber [5], Helmholtz resonator [7], cantilever beam resonator [8] and membrane with attached mass [9].
Nonetheless, although locally resonant metamaterials are applicable for manipulating wave propagation and providing low-frequency vibration attention, broad bandgaps are hard to be achieved by metamaterials. Few researchers presented the metamaterials with non-uniform resonators as a method of enlarging the width of bandgap. Sun et al. [10] and Pai [11] made the first attempt at investigating metamaterials with spatially varying mass-spring-damper subsystems. Their investigations proved that metamaterials with properly designed spatially varying local resonators can achieve broader bandgap than that with uniform resonators. Nonetheless, there are not enough existing research noticed the importance of non-uniform resonators, let alone proposed metamaterials with well designed spatially varying resonators.

In order to enlarge the bandgap of locally resonant metamaterials, an elastic metamaterial that constructed by Π-shaped beams with rainbow-shaped cantilever-mass resonators is developed in the present paper. An analytical model is first proposed to solve the frequency response function of the rainbow metamaterial. The analytical model is later validated by comparison with experimental results. After that a genetic algorithm optimization method is applied to search the optimal non-uniform distributions of resonator mass that can generate best receptance within described frequency range.

2. Analytical modelling method for the rainbow metamaterial

An analytical model is set up to determine the structural dynamics of the rainbow metamaterial. Figure 1 shows the schematic of the proposed rainbow metamaterial. The Π-shaped beam is partitioned into subspaces by periodic plate insertions. Cantilever-mass subsystems are clamped to the two side walls of the Π-shaped beam in each subspace.

![Schematic of rainbow metamaterial with spatially varying resonators](image1)

Figure 1. Schematic of rainbow metamaterial with spatially varying resonators

![Top view of two segments of the rainbow metamaterial](image2)

Figure 2. Top view of two segments of the rainbow metamaterial

As a Euler-Bernoulli beam, displacements of the Π-shaped beam inside the nth segment before and after the cantilever-mass can be written as,

\[ w_{i,n} = a_n e^{-j(x-x_{n,1})} + b_n e^{-j(x-x_{n,2})} + c_n e^{j(x-x_{n,1})} + d_n e^{j(x-x_{n,2})} \]

(1)

\[ w_{o,n} = a_n e^{-j(x+x_{n,1}+p_{n,0})} + b_n e^{-j(x+x_{n,2}+p_{n,0})} + c_n e^{j(x+x_{n,1}+p_{n,0})} + d_n e^{j(x+x_{n,2}+p_{n,0})} \]

(2)
where $k = (\rho A/EI_z)^{1/4}\sqrt{\omega}$, $A = w_d b_d + 2t_d H_d$ represents the cross-section area of the Π-shaped beam. $I_z$ is the cross section moment of inertia of the Π-shaped beam about its centroidal axis, given as

$$I_z \approx \frac{H_d^3 t_d^2 + 2t_d H_d w_d b_d}{3A} \tag{3}$$

The relation between the two displacements is derived as

$$\begin{bmatrix} \alpha_{n,r} \\ \beta_{n,r} \\ \chi_{n,r} \\ \epsilon_{n,r} \end{bmatrix} = R_c^{-1} R_n \Lambda_{n,l} \begin{bmatrix} \alpha_{n,l} \\ \beta_{n,l} \\ \chi_{n,l} \\ \epsilon_{n,l} \end{bmatrix} \tag{4}$$

where the matrices $R_c$, $R_n$ and $\Lambda_{n,l}$ are

$$R_c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -i & -1 & i & 1 \\ -1 & 1 & -1 & 1 \\ iEI_k^3 & -EI_k^3 & -iEI_k^3 & EI_k^3 \end{bmatrix}$$

$$R_n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -i & -1 & i & 1 \\ -1 & 1 & -1 & 1 \\ (iEI_k^3 + N_{s,n}) & (-EI_k^3 + N_{s,n}) & (-iEI_k^3 + N_{s,n}) & (EI_k^3 + N_{s,n}) \end{bmatrix}$$

$$\Lambda_{n,l} = \text{diag}(e^{i\phi_{p,n}L_d}, e^{-i\phi_{p,n}L_d}, e^{i\phi_{p,n}L_d}, e^{-i\phi_{p,n}L_d})$$

Displacement of the Π-shaped beam inside $(n+1)$th unit cell before the resonators is written as,

$$w_{n+1,j} = \alpha_{n+1,j} e^{i(x-x_n)} + \beta_{n+1,j} e^{-i(x-x_n)} + \chi_{n+1,j} e^{i(x-x_n)} + \epsilon_{n+1,j} e^{-i(x-x_n)} \tag{6}$$

Displacement recursion formulas between $n$th and $(n+1)$th unit cell is given as,

$$\begin{bmatrix} \alpha_{n+1,j} \\ \beta_{n+1,j} \\ \chi_{n+1,j} \\ \epsilon_{n+1,j} \end{bmatrix} = R^{-1} U \Lambda_{n,r} \begin{bmatrix} \alpha_{n,r} \\ \beta_{n,r} \\ \chi_{n,r} \\ \epsilon_{n,r} \end{bmatrix} \tag{7}$$

where $R$, $U$ are
\[
\Lambda_{nr} = \text{diag}(e^{-2((1-p)k)L}, e^{-2((1-p)k)L}, e^{2((1-p)k)L}, e^{2((1-p)k)L})
\]

\[
\mathbf{R} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-i & -1 & i & 1 \\
-\alpha L k^2 & \alpha L k^2 & -\alpha L k^2 & \alpha L k^2 \\
i\alpha L k^3 & -\alpha L k^3 & -i\alpha L k^3 & \alpha L k^3 \\
\end{bmatrix}
\]

\[
\mathbf{U} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-i & -1 & i & 1 \\
(-\alpha L k^2 + i J, \omega \phi k) & (\alpha L k^2 + J, \omega \phi k) & (-\alpha L k^2 - i J, \omega \phi k) & (\alpha L k^2 - J, \omega \phi k) \\
i\alpha L k^3 + m, \omega \phi & -\alpha L k^3 + m, \omega \phi & -i\alpha L k^3 + m, \omega \phi & \alpha L k^3 + m, \omega \phi \\
\end{bmatrix}
\]

Assuming a finite metamaterial is of free-free boundary and subjected to an excitation \( F \) at one end, given the equilibrium conditions, governing equations at the two ends are,

\[
F + m, \omega^2 w_{l=0} = EI_z \frac{\partial^3 w_{l=0}}{\partial x^3} \tag{9}
\]

\[
-J, \omega^2 w'_{l=0} = EI_z \frac{\partial w_{l=0}}{\partial x} \tag{10}
\]

\[
EI_z \frac{\partial^3 w_{x=L}}{\partial x^3} + m, \omega^2 w_{x=L} = 0 \tag{11}
\]

\[
EI_z \frac{\partial w_{x=L}}{\partial x} - J, \omega^2 w'_{x=L} = 0 \tag{12}
\]

The receptance function of the rainbow metamaterial is defined as,

\[
R_{rc} = 20 \log_{10} \left| \frac{w_{l=0}}{F} \right| \tag{13}
\]

3. Experimental validation

The analytical model is verified by comparing with the experimental results. The tested samples are fabricated by additive manufacturing method. The receptance function of printed samples are measured by receptance measuring system as shown in Fig. 3.
4. Optimization strategy

Since the frequency functions of the rainbow metamaterials are greatly affected by the distributions of resonator mass, we thus assign the resonator mass as design variables to achieve effective vibration attenuation by virtue of rational optimization strategy as shown in Fig. 5.

In order to maximize the vibration attenuation by the local resonators in a prescribed frequency range, two optimization strategies are proposed that invoke two objective functions individually.
One of the objective functions is set based on the maximum receptance value within the prescribed frequency ranges, written as

$$\min \ max\left( R_{\infty}(M_1, M_2, \Phi) \right)$$

(14)

where $M_1 = (m_{11}, m_{12}, \ldots, m_{it})$ and $M_2 = (m_{21}, m_{22}, \ldots, m_{2t})$ represent the mass of resonators at different sides of the complex beam, $\Phi = (f_1 \sim f_2)$ is the prescribed frequency regime. Receptance within the prescribed frequency range are destined to be low when the maximum value remains minimal, the vibration attenuation is thus optimized.

Constrains of the design variables are given as,

$$s.t. \quad m_{i1} = m_{2i},$$

$$m_{11}, m_{12}, \ldots, m_{it} \geq 0$$

$$2 \sum m_{ii} \leq 0.3M$$

$$i = 1, 2, \ldots, t$$

(15)

Average value is another evident assessment criteria of the receptance within a prescribed frequency range, therefore, the other objective function is set up based on the average receptance value within the prescribed frequency range, written as

$$\min \int_{\Phi} R_{\infty}(A, B, C, D, \Phi) df$$

$$f_2 - f_1$$

(16)

The constrains of design variables are the same as shown in Eq. (15). However, the integral is hard to achieve for complex receptance functions, the integral can be mathematically approximated by

$$\int_{\Phi} R_{\infty}(A, B, C, D, \Phi) df \approx \sum_{\Phi} R_{\infty}(A, B, C, D, \Phi) \Delta f$$

(17)

where $\Delta f$ denotes the frequency step.

The objective function hence becomes

$$\min \sum_{\Phi} R_{\infty}(A, B, C, D, \Phi) \Delta f$$

$$f_2 - f_1$$

(18)

5. Optimization example

Assuming the prescribed single frequency range is 130-150 Hz. As discussed in Sections 4, two objective functions are defined, hence two fitness functions are applicable for the GA optimization process as shown in Fig. 5. Receptance values of the two rainbow metamaterials with optimal resonator mass distributions are compared with that of complex beams of the same mass but without resonators in Figs. 6 and 7 respectively. The optimal rainbow metamaterials in Figs. 6 and 7 are obtained by the maximal receptance value based objective function and average value based objective function respectively. As it can be seen, both of the two optimal rainbow metamaterials show bandgaps within the prescribed frequency range, the receptance values are hence greatly reduced. The optimal metamaterial in Fig. 6 has a maximal receptance about 38 dB less than that of the structure without resonators, while the average receptance difference between the optimal structure and the no-resonator beam in Fig. 7 is about 33 dB,
that is, both maximum and average displacements within 130–150 Hz can be reduced by a factor of more than 70 with the optimization process.

Besides, it also can be seen from Figs. 6 and 7 that the optimal structure by maximum value based objective function has broader bandgap but bigger receptance value within the prescribed frequency range, which is opposite to the optimal structure derived by average value based objective function. Optimization strategy can be chosen according to requirements of different applications.

Figure 6. Receptance comparison between optimal rainbow metamaterial by maximal value based objective function and no-resonator complex beam

Figure 7. Receptance comparison between optimal rainbow metamaterial by average value based objective function and no-resonator complex beam

REFERENCES


