FUNDAMENTAL FREQUENCIES OF A CANTILEVERED NANOBEAM WITH ARBITRARY BOUNDARY CONDITIONS INCLUDING SURFACE EFFECTS.

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Abstract

This investigation is about the motion of a cantilevered beam used in dynamic atomic force microscopy (dAFM) which can be utilized in scanning the topographical features of biological samples or "pliable" samples in general. These cantilevered beams can be used to modify samples by using high frequency oscillations to remove material or shape nano structures. A cantilever nanobeam with arbitrary boundary conditions is studied to investigate different configurations and the effects on the relevant parameters.

The nano structure is modelled using the Euler-Bernoulli theory. Eringen's theory of non-local continuum is incorporated to simulate the dynamics of the system. This theory is effective at nano-scale because it takes into account the small-scale effects of the structure. The scanning process is achieved by tapping/contact with the sample surface to determine the topographical profile of the sample. The tapping contact force can also be used to deform the sample surface or remove material using high frequency oscillations.

The fixed end is modelled as a torsional spring with zero transverse displacement instead of the "ideal" or clamped boundary condition. The torsional boundary condition can be tuned, by changing the torsional spring stiffness, such that the compliance of the system matches that of the sample to prevent mechanical damage of both the cantilever tip and the sample. The boundary condition at the free end is a tip-mass attached to a transverse linear spring which models a contact force.

At nano scale, the surface area to bulk ratio increases and surface effects becomes a significant factor when determining the natural frequencies of the system. The motions of the tip of the beam and the tip-mass is investigated to frequency response and force. The tip response frequency includes information about the maximum displacement amplitude and therefore the sample penetration depth.

Keywords: atomic force microscope, tip mass, elastic restraints, vibrations, nanobeam, small scale effects.
1. Introduction.

The vibration characteristics of nano scale beams can be analysed within the several frameworks depending on the size of the beam [1,2]. At macro and micro level, the Euler-Bernoulli and Timoshenko bending theories are used with a satisfactory results. When the beams reach nanoscale length, non-local continuum (stress gradient) and strain gradient theories are incorporated in the modelling [3-8]. These stress gradient theories include Eringen’s non-local theory (first order stress) and higher order theories like Reddy Beam Theory (RBT) and Levison Beam Theory (LBT). Eringen’s theory provides a unified foundation for field equations of non-local continuum and provides a basis for several stress-gradient theories [9]. Higher order stress/strain gradient theories are constructed such that the transverse stress at the surface vanish as required. Ansari et al. [10] and Lu et. al. [11] investigated strain gradient theories for beams at nanoscale by taking into account only the local higher order strains. All these theories above provide very accurate results compared to Molecular Dynamics (MD) simulations.

According to literature, scientist and engineers are presently extending their investigations to probe the behaviour of beam at pico-meter (pm) scale [12,13]. At these scales, the beams under investigation could be at minimum, three (3) atoms thick , and the bulk to surface volumes become comparable such the surface energies have significant influence on the vibration of the system. This influence on the natural frequencies of vibration is borne out of the fact that the different layers of the beam experience different environments.

The bulk material is typically surrounded by other atoms, whilst the surface atoms are in contact with the bulk atoms on one side and a different “environment” on the other side (i.e. air or viscous fluid). In the present study Gurtin and Murdoch’s Linear Surface Elasticity Theory (LSET) is adopted to model the influence of surface effects on the system [14]. The system is modelled as a nanobeam with a torsional boundary condition at one end and spring-mass at the free end.

2. Elastically restrained nanobeam with spring–mass system at \( x = L \).

The nanobeam under investigation is restrained by a torsional spring to at \( x = 0 \) and is modelled as a flexible restraint, and with a single degree of freedom spring-mass system attached at the other end \( x = L \). Fig. (1a) shows the dAFM tip and Fig. (1b) shows the dAFM interacting with the sample.

Attached to the free end of the beam at \( x = L \) is a scanning tool which can be modelled as a mass with a sharp tip, attached to the beam by means of a linear spring \( (k_2) \) and the centre of gravity of the tip mass coincides with the tip of the beam and this constitutes the spring-mass system. During the scanning process, a contact force is generated between the tip mass and the object to be profiled. The spring-mass system is excited by the motion of the tip of the nanobeam as it vibrates and the constitutive relation of stress-strain for the beam based on nonlocal theory of elasticity can be expressed as

\[
\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E^e \varepsilon_{xx}
\]
where $E^e$ is the effective Young’s modulus including surface effects, $\bar{\mu} = e_o l_i$ is the small scale parameter with $e_o$ denoting a material constant and $l_i$ the characteristic length. The expression for moment $M(x)$ is given by

$$M(x) - \bar{\mu} \frac{\partial^2 M(x)}{\partial x^2} = E^e I^e \frac{\partial^2 w}{\partial x^2}$$

(2)

where $I^e$ is the effective moment of inertia. The equation of motion for a nonlocal nano beam undergoing transverse motion is given in (9) and can be expressed as

$$E^e I^e \frac{\partial^4 w(x,t)}{\partial x^4} - \tau_o s^* \frac{\partial^2 w(x,t)}{\partial x^2} + (\rho A + \rho_o s^*) \frac{\partial^4 w(x,t)}{\partial t^2} + \frac{2vl\rho_o}{H} \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + N \frac{\partial^2 w(x,t)}{\partial x^2} - \bar{\mu} N \frac{\partial^2 w(x,t)}{\partial x^2} \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} = F_o(x,t)$$

where,

$$E^e I^e = E I + (2\mu_o + \lambda_o) I^* - \frac{2vl\tau_o}{H} \frac{\partial^2 z(t)}{\partial x^2} k z(t) = k_2 w(L,t)$$

(3)

The Young’s modulus of the bulk is $E$, $\mu_o$ and $\lambda_o$ are the Lamme’s constant that can be determined using molecular dynamics MD. $I$ is the moment of inertia of the bulk and $I^*$ is the perimeter moment of inertia of the surface layer and $\nu$ is the Poisson's ratio whilst $\tau_o$ is the residual surface stress under unconstrained conditions. The density of the bulk is $\rho$ and $\rho_o$ that of the surface layer. $A$ is the cross-sectional area and $s^* = \int n z^2 ds$ is the cross-sectional area of the surface layer. $F_o(x,t)$ is the forcing function which is taken as $F_o(x,t) = 0$ for a beam under free vibration. The dynamic motion of the tip mass is expressed in Eq. (4) and the two Eqs. (3) and (4) are coupled through the motion of the tip $w(L,t)$.

Fig. 2. Section of beam showing forces and moments when beam is displaced from equilibrium position.

### 3. Method of solution.

Solution of the governing Eq. (3) is obtained by eigenfunction expansion of the displacement function as

$$w(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

(5a)

$$z(L,t) = \sum_{n=1}^{\infty} z_n T_n(t)$$

(5b)

$$\ddot{T}_n(t) + \omega_n^2 T_n(t) = 0$$

(5c)

Inserting Eq. (5a) into Eq. (3) and using Eq. (5c), we obtain the differential equation of motion for the beam in the modal domain. For the spring-mass, after inserting Eq. (5b) into Eq. (4) and using Eq. (5c), we obtain the equation of motion of the tip mass in the modal domain. The natural frequency $\omega_n$ is the frequency of the $n^{th}$ mode of vibration. The differential equation in the modal mode then becomes

$$P \cdot X'''' + Q \cdot X'' - R \cdot X = 0$$

(6a)

where,

$$P = 1 + \left[\frac{(2\mu_o + \lambda_o)}{E} - \frac{2v \tau_o}{\alpha H} \frac{a}{\beta^2} \right]$$

and $\mu = e_o l_i$.
\[ Q = \frac{2v}{A} \alpha \rho_c a_n^4 + \mu \left(1 + \rho_c \frac{s^*}{A}\right) a_n^4 + \beta^2 - \tau_c \frac{s^*}{I}, \]

\[ R = \left(1 + \rho_c \frac{s^*}{A}\right) a_n^4 \]

and,

\[ \alpha = \frac{l'}{l} H, \quad \tau_c = \frac{\tau_0}{\varepsilon l}, \quad \mu = \frac{\mu_o}{\varepsilon l}, \quad \beta^2 = \frac{N}{E l^2}, \quad \rho_c = \frac{\rho_c}{\rho_c}. \]

The motion of the tip-mass can also be written in the modal domain and the modal displacement of the system is

\[ z(L, t) = \frac{X(L)}{1 - a_n^4/a_k^4} \]

where \( a_k^4 = \kappa_2/\eta \) is frequency parameter for the spring-mass system; \( \kappa_2 \) and \( \eta \) represent the linear spring constant ratio and the tip-mass ratio, respectively.

The frequency parameter \( a_k \) is associated with the natural frequency \( \omega_n \) of vibration, \( \alpha \) is a constant ratio that depends on the moment of inertia of the cross-section geometry of the bulk and surface, \( \tau_c \), \( \mu \), \( \beta^2 \), and \( \rho_c \) are the dimensionless constants corresponding to the residual stress, small-scale parameter, axial load and bulk to surface density ratio, respectively. The general solutions of Eq. (3) is given by

\[ X_n(x) = A_n \cos p_{2n} + B_n \sin p_{2n} + C_n \cosh p_{1n} + D_n \sinh p_{1n} \]

where \( p_{1n} \) and \( p_{2n} \) are

\[ p_{1n} = \sqrt{Q + \sqrt{Q^2 + 4PR}} \quad \text{and} \quad p_{2n} = \sqrt{Q - \sqrt{Q^2 + 4PR}} \]

The constants \( A_n \), \( B_n \), \( C_n \) and \( D_n \) are determined from the boundary conditions where, the boundary conditions at \( x = 0 \) are zero displacement and moment and can be expressed as

\[ w(0) = 0 \]

\[ E \alpha \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{2v}{l} \rho_o \frac{\partial^2 w(x,t)}{\partial t^2} + \bar{\mu} \left[N \frac{\partial w(x,t)}{\partial x^2} - (\rho A + \rho_o s^*) \frac{\partial^2 w(x,t)}{\partial t^2} - (2\mu_o + \lambda_o) \frac{\partial^2 w(x,t)}{\partial x^2} \right] = 0 \]

where \( k_1 \) is the torsional spring constant. Using Eqs. (5a) and (5b) the moment boundary condition can be transformed and written in the modal mode in Eqs. (9a) and (9b), where \( \kappa_1 \) the torsional is spring constant ratio.

\[ X(0) = 0 \]

\[ \frac{(2\mu_o + \lambda_o) l'}{l} \frac{X'''}{X} + \left[1 + \frac{(2\mu_o + \lambda_o)}{E} \frac{\alpha}{l} \right] - \mu \beta^2 + \frac{2v}{l} \rho_o a_n^4 + \tau_c \frac{s^*}{I} \right] X'' + a_n^4 \left[ \frac{(2\mu_o + \lambda_o) l'}{l} \frac{X''}{X} + \mu \right] \]

At the free end, taking into account the small scale effect, tip mass and the linear spring, the shear boundary condition at \( x = L \) can be expressed as

\[ E \alpha \frac{\partial^2 w(x,t)}{\partial x^3} + \tau_o s^* \frac{\partial w(x,t)}{\partial x} + \frac{2v}{l} \rho_o \frac{\partial^2 w(x,t)}{\partial x^3} - N \frac{\partial w(x,t)}{\partial x} + \frac{\partial^2 w(x,t)}{\partial x^3} = F_L(x,t) \]

where \( F_L(x,t) = k_2 \cdot z(x,t) \) is force due to the spring-mass system and \( k_2 \) is the linear spring constant. Equation (6b) is the solution to Eq. (4) is used to represent \( z(L,t) \) in the shear boundary. Using Eqs. (5a) and (5b) the shear boundary condition Eq. (10a) can be transformed and written in the modal mode in Eq. (10b),
\begin{align}
\mu \frac{(2\mu_o+\lambda_o) I^*}{E} \frac{X'''}{\tau} + \left[ \left( 1 + \frac{(2\mu_o+\lambda_o)}{E} - \frac{2\nu \tau c}{\alpha} \right) \frac{A}{\alpha} + \mu \left( \beta^2 + \frac{\tau c}{\alpha} \frac{s^*}{I} - \frac{2\nu I^*}{A} \right) \right] X'' + a_n^4 \mu \left( 1 + \frac{\rho c}{\alpha} \frac{s^*}{\tau} \right) \alpha \beta^2 X' - \frac{2\nu \tau c}{\alpha} \frac{s^*}{\tau} \frac{A}{\alpha} \frac{\beta^2}{\mu} X = 0
\end{align}

Substitution of Eq. (7a) into the boundary condition Eq. (9a), we obtain

\[ A_n + C_n = 0 \]

The general solution Eq. (7a) can now be expressed as

\[ X_n(x) = B_n \sin \frac{\pi n}{L} + D_n \sinh \frac{\pi n}{L} + \left( \cos \frac{\pi n}{L} - \cosh \frac{\pi n}{L} \right) C_n \]

substituting Eq. (12) into Eq. (8b) gives

\[ B_n = \frac{C_n}{\Gamma_1} - \frac{p_{1n}^2(2\mu_o+\lambda_o) I^*}{p_2 E} \frac{p_{2n}^2}{p_2} + \frac{p_{1n}^2}{p_2} + \frac{p_{2n}}{p_2} D_n \]

and substituting \( B_n \) in Eq. (12) into Eq. (11) gives the general solution expressed in terms of constants \( C_n \) and \( D_n \) alone. This result can be substituted into the moment boundary conditions Eq. (9a) at \( x = L \) to obtain

\[ C_n \cdot \Gamma_1 + D_n \cdot \Gamma_2 = 0 \]

were \( \Gamma_{1n} \) and \( \Gamma_{2n} \) are the constants extracted from applying the moment boundary condition, Eq. (9a) or (9b). After substituting the results of Eq. (4) and (13) into the shear boundary condition Eqs. (10a) or (10b) at \( x = L \) can be expressed as

\[ C_n \cdot \Gamma_3 + D_n \cdot \Gamma_4 = 0 \]

were \( \Gamma_{3n} \) and \( \Gamma_{4n} \) are the constants extracted from applying the shear boundary condition, Eq. (9a) or (9b) at the free end \( x = L \). The results from the moment and shear boundary conditions given in Eqs. (14) and (15) can be expressed in matrix form as

\[ \begin{bmatrix} \Gamma_{1n} & \Gamma_{2n} \\ \Gamma_{3n} & \Gamma_{4n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

and the characteristic equation can be obtained from the determinant of Eq. (16) as:

\[ \Gamma_{1n} \cdot \Gamma_{4n} - \Gamma_{2n} \cdot \Gamma_{3n} = 0 \]

The characteristic Eq. (17) can be solved numerically to compute the roots where \( \kappa_1, \kappa_2, \eta, \eta, a_n^4, \beta^2, \mu \) and \( a_k^4 \) are the dimensionless constants for the beam and the spring-mass system.

**4. Frequency equations for arbitrary boundary conditions.**

The structure shown in Fig. (1), indicate a torsional spring with torsional spring constant \( \kappa_1 \) at \( x = 0 \). When the spring constant approaches zero \( (\kappa_1 \to 0) \), the restraint at \( x = 0 \) behaves as that of a pin support where there is zero resisting moment and the beam is allowed to spin freely. When the spring constant is not zero, there is a resisting moment at the boundary and therefore the torsional spring has an effect on the vibrations of the system. At the extreme end the spring constant approaches infinity \( (\kappa_1 \to \infty) \). In this case the support is completely rigid and the boundary condition is that of a cantilevered beam. In Eq. (12), when \( \kappa_1 \to \infty \) the first two terms vanish in \( B_n \) above, and consequently in the characteristic equation too.

At \( x = L \), a tip mass is attached to the beam by means of a transverse linear spring. When the linear spring constant is zero \( (\kappa_2 \to 0) \) the effect of the tip mass is not realized at the tip of the beam and therefore the tip-mass has no influence on the natural frequencies.
Table 1: Classical boundary condition derived from the system.

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clamped-Free (CF)</td>
<td>$\infty$</td>
<td>0 or $\infty$</td>
<td>0</td>
</tr>
<tr>
<td>Simply supported-Free (SF)</td>
<td>0</td>
<td>0 or $\infty$</td>
<td>0</td>
</tr>
<tr>
<td>Simply supported</td>
<td>0</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Clamped-Simply supported</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

However, as the linear spring constant increases ($\kappa_2 \to \infty$), the effect of the tip-mass become pronounced and when $\kappa_2$ is extremely large, the tip-mass is rigidly attached to the tip of the beam. By varying the torsional spring constant, linear spring constant and tip-mass ratio, we can derive the classic frequency equation for the total system including torsional spring and spring-mass is given below in Eq. (21). The following frequency equations are derived from the boundary conditions in Table 1.

**Clamped-Free (CF) with radial axial load:**

$$\sinh(p_1 L) \sin(p_2 L) \left(\frac{2\beta^2}{p_1 p_2} - \frac{p_2}{p_1} + \frac{p_1}{p_2}\right) + \cosh(p_1 L) \cos(p_2 L) \left(\beta^2 \left(\frac{1}{p_2^2} - \frac{1}{p_1^2}\right) - \beta^2 \left(\frac{1}{p_2^2} + \frac{1}{p_1^2}\right) - \frac{p_2^2}{p_1^2} - \frac{p_1^2}{p_2^2}\right)$$

(18)

**Simply supported-Free (SF):**

$$\frac{1}{\kappa_1} \left(\frac{p_2 + \frac{p_1^2}{p_2}}{p_1} \sinh(p_1 L) \sin(p_2 L) + \left(-\frac{p_2}{p_1} - p_1\right) \sinh(p_1 L) \cos(p_2 L)\right) + \left(p_1 - \frac{p_2}{p_1}\right) \sinh(p_1 L) \sin(p_2 L) - 2 \cosh(p_1 L) \cos(p_2 L) - \frac{p_2^2}{p_1^2} - \frac{p_1^2}{p_2^2}$$

(19)

**Clamped-Simply supported (CS):**

$$\frac{1}{\kappa_1 \eta} \left(-\frac{p_1 p_1^2 \cosh(p_1 L) \sin(p_2 L)}{a_1^4} - \frac{p_1^2 p_2 \cosh(p_1 L) \sin(p_2 L)}{a_2^4} + \frac{1}{\kappa_1 \eta} \left(\frac{p_2^4 \sinh(p_1 L) \cos(p_2 L)}{a_1^4} + \frac{1}{\kappa_1 \eta} \left(\frac{p_2^4 \sinh(p_1 L) \cos(p_2 L)}{a_2^4} - \frac{p_1^2 p_2 \cosh(p_1 L) \sin(p_2 L)}{a_1^4} \right) + \frac{1}{\eta a_1^4} \left(p_2^4 - \frac{p_1^2}{p_2} p_2 \sinh(p_1 L) \sin(p_2 L) + 2 p_1 p_2^2 \cosh(p_1 L) \cos(p_2 L) + \frac{p_1^4}{p_2} + p_1^3\right) + \frac{1}{\eta a_2^4} \left(p_2^2 p_2 - \frac{p_1^2}{p_2} p_2 \sinh(p_1 L) \sin(p_2 L) - 2 p_1 p_2^2 \cosh(p_1 L) \cos(p_2 L) - \frac{p_1^4}{p_2} - p_1^3\right) + \frac{1}{\kappa_1} \left(p_1^2 \sinh(p_1 L) \sin(p_2 L) + 2 p_2 \sinh(p_1 L) \sin(p_2 L) + \frac{p_2^4}{p_1} \sinh(p_1 L) \cos(p_2 L)\right) + \left(p_2 + \frac{p_1}{p_2}\right) \cosh(p_1 L) \sin(p_2 L) - \left(\frac{p_2^2}{p_1^2} + 1\right) \sinh(p_1 L) \cos(p_2 L)$$

(20)

**Torsional cantilever with spring mass system:**

$$\frac{1}{\kappa_1} \left(\cosh(p_1 L) \sin(p_2 L) \left(\frac{p_2^2 a_1^2 \beta^2}{p_1^3 a_1^4} - \frac{p_2^2 \beta^2}{p_1^4} + \frac{p_2 a_1^2 \beta^2}{p_1^3 a_1^4} - \frac{p_2^2 \beta^2}{p_1^4} - p_2^2 a_1^4 + p_2 a_1^4 - a_1^4 - a_1^4\right)\right) + \frac{1}{\kappa_1} \left(\sinh(p_1 L) \cos(p_2 L) \left(\frac{p_2^2 a_1^2 \beta^2}{p_1^3 a_1^4} - \frac{p_2^2 \beta^2}{p_1^4} + \frac{p_2 a_1^2 \beta^2}{p_1^3 a_1^4} - \frac{p_2^2 \beta^2}{p_1^4} - p_2^2 a_1^4 + p_2 a_1^4 + a_1^4 - a_1^4\right)\right) + \left(\sinh(p_1 L) \cos(p_2 L) \left(-\frac{2 p_2 a_1^2 \beta^2}{p_1^3} - \frac{p_2 a_1^2}{p_1^4} + \frac{p_2 a_1^2}{p_1^4}\right) + \sinh(p_1 L) \sin(p_2 L) \left(-\frac{2 p_2 a_1^2 \beta^2}{p_1^3} + \frac{p_2 a_1^2}{p_1^4} - \right)\right)$$

(21)
\( \frac{p_2}{p_1^2} \) 
\[ + \frac{p_2^2 \beta^2}{p_1^4} + \frac{1}{\kappa_1} \left( \sinh(p_1 L) \sin(p_2 L) \left( \frac{p_2^2 \eta a_4}{p_1^2} + \frac{2p_2 \eta a_4}{p_1^2} + \frac{\eta a_4^2}{p_1^2 p_2} \right) + \right. \\
\left. \sinh(p_1 L) \cos(p_2 L) \left( -\frac{p_2^2 \eta a_4}{p_1^2} - \frac{\eta a_4^2}{p_1^2 p_2} \right) + \cosh(p_1 L) \sin(p_2 L) \left( \frac{p_2^2 \eta a_4}{p_1^2} + \frac{\eta a_4^2}{p_1^2 p_2} \right) + \frac{p_2^4}{p_1^4} + 1 \right) \]  
(21)

The natural frequencies of the system \( R_n \) are obtained by making a substitution, \( a_n = R_n / L \), into the into Eq. (17). The values of \( R_n \) are dimensionless natural frequencies used below to analyse the numerical results. When \( \kappa_1 \rightarrow \infty \) the torsional spring becomes rigid and the boundary condition behaves like that of a cantilevered beam. The classic cantilever configure can be obtained by setting the mass to zero (\( \eta = 0 \)) and the fundamental natural frequency of the system is \( R_1 = 1.8750 \) which corresponds to the results obtained by Magra [15] \( R_1 = 0.5969\pi \). Furthermore, when \( \kappa_1 \rightarrow \infty \) the linear spring is rigid and the system behaves like a cantilevered beam with concentrated tip mass because the centre of gravity of the attached mass coincides with the tip of the beam (see Fig. (1)).

![Fig. 3. Fundamental frequency plotted against spring constants \( \kappa_1 \) and \( \kappa_2 \) with tip mass ratio: a) \( \eta = 0 \), 0.1, 0.5 and 1.](image)

For a tip mass ratio \( \eta = 0.1 \) and \( \eta = 1 \) the fundamental natural frequency is \( R_1 = 1.7227 \) and \( R_1 = 1.2479 \) which are the same result obtained by Magrab (\( R_1 = 0.5484\pi \)) and Gürgöze (\( R_1 = 1.2479 \)), respectively [15,16]. In practical applications, the values for the parameters of interest will vary between zero and infinity (\( 0 < \kappa_1, \kappa_2, \eta < \infty \)). The boundary conditions discussed above relate to extreme rigidity of the torsional and linear spring. It is also noted that the when the torsional spring is completely elastic \( \kappa_1 \rightarrow 0 \) the fundamental natural frequencies approach zero in the limit \( R_1 \rightarrow 0 \) and the other variables have minimal effect on the frequencies as seen in Fig. (1) above.

### 5. Conclusions.

In the present paper, small scale and surface effects on the fundamental frequency are investigated for a nanobeam with elastically restrained end conditions and carrying a tip mass attached via a linear spring to the end of the beam. The solution for the beam is obtained analytically by expanding the deflection in terms of its eigenfunctions and solving the resulting characteristic equation numerically. Furthermore, the characteristic equations are presented for parametric studies of the effect of support elasticity and tip mass on the fundamental frequencies of the nanobeam.

It is observed that the boundary conditions may lead to an increase or decrease of the fundamental frequency depending on the support flexibility. Boundary conditions can be expressed in terms of a torsional spring at \( x = 0 \), linear spring and tip mass at \( x = L \). The classical boundary conditions correspond to setting the torsional and linear spring constants to zero (\( \kappa_{1,2} \rightarrow 0 \)) or infinity (\( \kappa_{1,2} \rightarrow \infty \)).
It was observed that low torsional spring stiffness leads to a decrease in the fundamental frequency and high torsional spring stiffness to an increase in the fundamental frequency as the small scale parameter increases. The rates of decrease and increase depend on the relative values of the spring constants. The effect of the tip mass on the frequencies is to lower the natural frequencies as observed in [17] and [18].

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7. REFERENCES