This paper investigates the dynamic behaviour of a rotordynamic system incorporating stiffness non-linearity in the supporting bearings. The system model comprises a horizontal Jeffcott rotor mounted on symmetrically located ball bearings. The nonlinear radial stiffness of the bearing is estimated based on the Hertzian contact theory. The harmonic balance approximations and the time-marching method are applied to solve the governing equations and determine the steady-state dynamic response. The results show that a small nonlinear bearing stiffness provides benefits for the suppression of response in the high frequency range. A large damping coefficient in the bearing can reduce the peak response amplitude. A combination of a small bearing stiffness and a large bearing damping can assist in effective vibration suppression from rotor in a broad excitation frequency range. These results lead to a better understanding of the effects of stiffness nonlinearity and damping property of the bearing on the vibration behaviour of rotordynamic system and hence benefit enhanced designs with desirable dynamic characteristics.

Keywords: rotordynamics, nonlinear bearing, harmonic balance method, nonlinear stiffness

1. Introduction

There have been much recent interest in understanding nonlinearities in engineering vibration systems and the use of nonlinear elements for vibration suppression [1-6]. Rotor-bearing dynamic systems are widely applied in rotating machinery such as electric motor/generator and aero-engines. Much efforts have been made on accurate vibration modelling and suppression design of such systems. In practice, there is unavoidable unbalanced mass arising from rotor which acts as the vibration source. The rotor system with rolling element bearing support may exhibit nonlinear behaviour due to the Hertzian non-linear contact force, bearing clearance, rolling element centrifugal force, lubrication, and etc [7]. Researchers have intensively investigated the effect of bearing nonlinearity on the dynamic behaviour of rotor-bearing systems. For instance, Tiwari et al. [8] examined the nonlinear behaviour of unbalanced rigid rotor horizontally supported on ball bearings with Hertzian contact and radial internal clearance by using Poincare map and interpolation technique. The bifurcations of rotor orbit were found, and chaotic motions of rotor were observed including period doubling and mechanism of intermittency. Chen [9]
investigated the effects of radial clearance and Hertzian contact force within the ball bearing on the unbalanced rotor bearing system by numerical integration method. Bai et al. [10] studied the flexible rotor supported by ball bearings with two degree-of-freedom element model and simulated the nonlinear response using the finite element approach. The effect of Hertzian force and bearing clearance on the resonance frequency of the system were studied and the numerical simulation results were also validated by the experimental work. Li et al. [11] analysed misaligned rotor system with ball bearing support by harmonic balance method, the Floquet theory was applied to obtain the motion stability of the rotor system at different rotating speed. The period doubling bifurcation and secondary Hopf bifurcation were obtained.

In this paper, the dynamic behaviour of a rotordynamic system incorporating stiffness nonlinearities and linear damping in the supporting ball bearings is investigated. The system model comprises a horizontal rigid Jeffcott rotor mounted on symmetrically located ball bearings. The nonlinear radial stiffness of the bearing is approximated by the Hertzian contact theory. The harmonic balance approximations with alternating frequency technique (AFT) and the numerical integration method are applied to solve the governing equations and obtain the steady-state dynamic response. The effects of the nonlinear bearing radial stiffness and linear bearing damping on the steady-state rotor response amplitude are studied. These results provide a better understanding of the influence of stiffness nonlinearities and damping properties within the bearings on the vibration characteristics and benefit the dynamic designs for improved performance.

2. Rotor bearing system model with unbalanced mass

A symmetric Jeffcott rotor model is shown in Fig. 1, with assumption that its mass \( m \) concentrates on the centre disk and the shaft is rigid. The mass eccentricity \( e \) between the shaft center and mass centre acts as the excitation source in the system. Two identical deep groove ball bearings are installed on two ends of the rotor with nonlinear radial stiffness and linear damping. The bearing restoring force acts in the radial directions. The rotor motion can be projected in \( x \) and \( y \) axis and the displacement response are denoted by \( x \) and \( y \), respectively. The axial motion along the \( z \) axis as well as rotational inertia of the rotor are not considered in this paper.

![Figure 1. A schematic representation of a Jeffcott rotor model with nonlinear bearing support.](image)

Based on the Hertzian theory and the point contact between the bearing ball and race in the ball bearing [1, 5, 6], the bearing contact force \( f_r(r, \dot{r}) \) can be simplified as expression of below
\[ f_r(r, \dot{r}) = kr^3, \quad (1) \]

where \( k \) represents the contact stiffness and \( r = \sqrt{x^2 + y^2} \) is the displacement of the centre of mass of the rotor in the radial direction.

Then the bearing restoring force projected in \( x \) and \( y \) direction shown in Fig. 1(b) could be obtained

\[ f_x = k \frac{r^3}{r} \dot{x} + c \dot{x} = kxr^2 + c \dot{x}, \quad (2a) \]
\[ f_y = k \frac{r^3}{r} \dot{y} + c \dot{y} = kyr^2 + c \dot{y}, \quad (2b) \]

where \( c \) is the bearing damping coefficient in \( x \) and \( y \) directions. Therefore, the equations of motion of the rotor-bearing system are

\[ m \ddot{x} + f_x = me\omega^2 \cos \omega t - mg, \quad (3a) \]
\[ m \ddot{y} + f_y = me\omega^2 \sin \omega t, \quad (3b) \]

where \( m \) is the rotor mass, \( x \) and \( y \) are the rotor displacement in \( x \) and \( y \) direction, respectively, \( e \) is the mass eccentricity, \( \omega \) is the rotor angular speed and \( mg \) is the gravity of the rotor. Those nonlinear equations can be solved by harmonic balance method with AFT [12].

### 3. Rotor response

To implement the harmonic balance method to obtain periodic responses, the steady-state response of the rotor in \( x \) and \( y \) directions are represented by harmonic terms with the fundamental oscillating frequency the same as the excitation

\[ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_{01} + \sum_{n=1}^{N} (A_{nc1}\cos n\tau + A_{ns1}\sin n\tau) \\ A_{02} + \sum_{n=1}^{N} (A_{nc2}\cos n\tau + B_{ns2}\sin n\tau) \end{bmatrix}, \quad (4a) \]
\[ \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{N}(-n\omega A_{nc1}\sin n\tau + n\omega A_{ns1}\cos n\tau) \\ \sum_{n=1}^{N}(-n\omega A_{nc2}\sin n\tau + n\omega A_{ns2}\cos n\tau) \end{bmatrix}, \quad (4b) \]
\[ \begin{bmatrix} \dddot{x} \\ \dddot{y} \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{N}(-(n\omega)^2 A_{nc1}\cos n\tau - (n\omega)^2 A_{ns1}\sin n\tau) \\ \sum_{n=1}^{N}(-(n\omega)^2 A_{nc2}\cos n\tau - (n\omega)^2 A_{ns2}\sin n\tau) \end{bmatrix}, \quad (4c) \]

where \( A_{01}, A_{nc1}, A_{ns1} \) and \( A_{02}, A_{nc2}, A_{ns2} \) provide the amplitudes and phases of response components and \( \tau = \omega t \). The nonlinear force term in the governing equation is also expressed by using harmonic terms

\[ \begin{bmatrix} \hat{f}_x \\ \hat{f}_y \end{bmatrix} = \begin{bmatrix} S_{01} + \sum_{n=1}^{N} (S_{nc1}\cos n\tau + S_{ns1}\sin n\tau) \\ S_{02} + \sum_{n=1}^{N} (S_{nc2}\cos n\tau + S_{ns2}\sin n\tau) \end{bmatrix}, \quad (5) \]

where \( S_{01}, S_{nc1}, S_{02}, S_{nc2}, S_{ns1} \) and \( S_{ns2} \) are the Fourier coefficients expressed by

\[ S_{01} = \frac{1}{2\pi} \int_{0}^{2\pi} f_x \cos \tau \, d\tau, \quad S_{02} = \frac{1}{2\pi} \int_{0}^{2\pi} f_y \cos \tau \, d\tau, \]
\[ S_{nc1} = \frac{1}{\pi} \int_{0}^{2\pi} f_x \cos n\tau \, d\tau, \quad S_{ns1} = \frac{1}{\pi} \int_{0}^{2\pi} f_y \cos n\tau \, d\tau, \]
\[ S_{nc2} = \frac{1}{\pi} \int_{0}^{2\pi} f_y \cos n\tau \, d\tau, \quad S_{ns2} = \frac{1}{\pi} \int_{0}^{2\pi} f_y \cos n\tau \, d\tau. \quad (6a-f) \]
For simple nonlinear functions, it may be possible to derive analytical expressions of the Fourier coefficients. Here the AFT scheme is used for numerical determination. Its idea is to use discrete Fourier transform (DFT) instead of Fast Fourier transform (FFT). Analysis procedures have been illustrated in previous work as ref. [12]. The nonlinear forces are obtained by

\[ \{ f_x \} = \{ \text{FFT}[ f_x \text{iFFT}[\{A_{101}, A_{nc1}, A_{ns1}\}], \text{iFFT}[\{-n\omega A_{nc1}, n\omega A_{ns1}\}]] \} \]
\[ \{ f_y \} = \{ \text{FFT}[ f_y \text{iFFT}[\{A_{202}, A_{nc2}, A_{ns2}\}], \text{iFFT}[\{-n\omega A_{nc2}, n\omega A_{ns2}\}]] \} \]

where FFT and iFFT denote one-dimensional forward and inverse discrete fast Fourier transform.

Eqs. (3) may be transformed into a matrix form as below:

\[ \{ P \} = \begin{bmatrix} m \ddot{x} \\ m \ddot{y} \end{bmatrix} + \begin{bmatrix} \{ f_x \} - \{ m\omega^2 \cos \omega t - mg \} \\ \{ f_y \} - \{ m\omega^2 \sin \omega t \} \end{bmatrix}. \]

(8)

By inserting Eqs. (4) and (5) into Eqs. (8) and set the Fourier coefficient of \( P \) to zero (truncated to order \( H \)), we obtain \( 2(2H + 1) \) nonlinear algebraic equations:

\[ P_0(A_0, A_1, ..., A_H) = 0, \]
\[ ... \]
\[ P_H(A_0, A_1, ..., A_H) = 0. \]

(9)

Then the Eqs. (9) can be simplified as:

\[ P(A) = 0, \]

(10)

where \( P = [P_0 \ P_{1c} \ P_{1s} ... \ P_{HS}]^T \), \( A = [A_0 \ A_{1c} \ A_{1s} ... \ A_{HS}]^T \) (T represents transpose operation) and each element in the matrix contains two degrees of freedom.

To solve the equation, \( P(A) = 0 \), Newton-Raphson method can be applied as below

\[ A^{j+1} = A^j - \left( \frac{\partial P}{\partial A} \right)^{-1}_{A(j)} P(A(j)), \]

(11)

where \( \frac{\partial P}{\partial A} \) is the Jacobian matrix and can be calculated by finite differences method, \( j = 0, 1, 2 ... H \) is the current harmonic order. Consequently, by combining the process of harmonic balance and AFT, the iterations can yield the solution of Fourier coefficients with proper accuracy.

Fig. 2 investigates the effects of the nonlinearities in the radial ball element bearing on the dynamic response of the Jeffcott rotor. Fig. 2(a) and (b) examines the effects of stiffness nonlinearity by keeping the damping of the bearing to be fixed with \( c = 0.02 \text{ Ns/m} \). Four different values of nonlinear stiffness for the bearing are considered with \( k = 0.1, 1, 2 \) and \( 3 \text{ N/m} \) for increasing bearing stiffness characteristics, respectively. The other system parameters are set to be \( m = 1 \text{ kg}, e = 0.1 \text{ m} \). The first-order HB-AFT results are represented by different lines while the numerical integration results are obtained by using the fourth-order Runge-Kutta method and denoted by symbols. Fig. 2(a) and (b) shows the use of nonlinear stiffness bearing bends the frequency-response curve of the rotor to the high-frequency range while the gravity of the rotor bends the response curve of \( x \) direction to low frequency range. When the ball bearing has higher stiffness parameter \( k \), the frequency response curves of the rotor shift more to the high frequencies. Multiple solutions can be found near the peak of the curve, implying the complex dynamic behaviour including bifurcations of the rotor. It also shows that the peaks of frequency response curves become higher with the increasing bearing stiffness. Fig. 2(c) and (d) examines the effects of damping magnitude by keeping the stiffness of the bearing to be fixed with \( k = 1 \text{ N/m} \). Four different values of damping for the ball bearing are considered with \( c = 0.02, 0.04, 0.06 \) and \( 0.08 \text{ Ns/m} \) for different linear damping characteristics, respectively. The other system parameters are set to be \( m = 
Comparing with the response with low damping ($c = 0.02$ Ns/m), the increasingly bearing damping will reduce the peak value of response curve of frequency response in $x$ and $y$ direction. The multiple solution range of the rotor can be minimized due to the increasing damping. Those results in Fig. 2 reveal that the damping and nonlinear stiffness of the bearing significantly affect the steady state response of the rotor.

Figure 2: Effects of stiffness and damping of bearing on the steady state response amplitudes, (a) and (c) for response in the $x$ direction while (b) and (d) for response in the direction of $y$. In (a) and (b), the solid, dashed, dotted lines and dash-dot line for $k_1 = 0.1, 1, 2$ and 3 N/m, respectively. In (c) and (d), the solid, dashed, dotted lines and dash-dot line for $c = 0.02, 0.04, 0.06$ and 0.08 Ns/m, respectively.

Fig. 3 investigates the effects of different combinations of rolling element ball bearing stiffness nonlinearities and damping levels on the dynamic response of the rotor. Four different combination values of nonlinear stiffness and linear damping for the bearing are considered with $k_1 = 0.1$N/m $c = 0.04$ Ns/m; $k_1 = 0.1$N/m $c = 0.08$ Ns/m; $k_1 = 1$N/m $c = 0.04$ Ns/m and $k_1 = 1$ N/m, $c = 0.08$ Ns/m, respectively. The other system parameters are set to be $m = 1$ kg, $e = 0.1$m. The first-order HB-AFT results are represented by different lines while the numerical integration results are obtained by
using the fourth-order Runge-Kutta method and denoted by symbols. As a comparison, Fig. 3 shows a combination of low stiffness and high damping within the bearing will suppress the steady state response of the Jeffcott rotor in the high-frequency range. This behaviour may be important for high speed rotor structure design and application.

![Figure 3: Effects of bearing stiffness and damping on the response, (a) and (b) for responses in the directions of x and y, respectively. The solid, dashed, dotted and dash-dot lines for $k_1 = 0.1\,\text{N/m}, c = 0.04\,\text{Ns/m}; k_1 = 0.1\,\text{N/m}, c = 0.08\,\text{Ns/m}; k_1 = 1\,\text{N/m}, c = 0.04\,\text{Ns/m}$ and $k_1 = 1\,\text{N/m}, c = 0.08\,\text{Ns/m}$, respectively.]

**Conclusion**

The dynamic behaviour of a rotordynamic system incorporating stiffness nonlinearities and linear damping in the supporting ball bearings have been investigated. The effects of the nonlinear bearing radial stiffness and linear damping on the steady state rotor response amplitude are examined. It was found that with the increase of the stiffness parameter, the response curves for the rotor shift to the high-frequency range with possible bending of the response peaks. A combination of low stiffness and high damping within the bearing was found to be beneficial for the suppression of the steady-state response of the rotor in a broad frequency range. These preliminary results provide some better understanding of the influence of stiffness nonlinearities of bearing supports on vibration characteristics of the rotordynamic system. Further work will be done on more detailed analysis with the aim of achieving enhanced rotordynamic system designs for improvement in dynamic performance.

**REFERENCES**


