USING CRAMER-RAO BOUND TO EVALUATE THE FAR-FIELD SOUND FIELD CONTROL EFFECT OF THE LOW-FREQUENCY PHASED ARRAY EMITTING THE DIFFERENT WEIGHT VECTORS

LI Haifeng, Ma Li, Zeng Juan, LIU Haijun, PENG Dayong, ZHAO Wenyao

Key Laboratory of Underwater Acoustic Environment, Institute of Acoustics, Chinese Academy of Sciences, Beijing, China 100190

e-mail: cust77@mail.ioa.ac.cn

The theory and simulation studies were developed to analyze the far-field sound field control effect of the low-frequency phased array emitting the different weight vectors. Firstly, combining the adiabatic normal mode theory, the sonar-target model excited by the phased array was derived. Based on the model, the relationship between the weight vector and the Cramer-Rao lower bounds (CRB) of time-delay and Doppler frequency-shift estimations were set up theoretically. Secondly, using the CRBs to evaluate the far-field sound control effect of the phased array emitting the several weight vectors were computed according to the simulated environment. Finally, under the same energy input conditions, from the simulating results, it was showed that the phased array emitting the No.1 normal mode was not always the optimum in the far-field sound field control.

1. Introduction

The underwater low-frequency phased linear array is the key hardware equipment of the single-mode excitation. It's the newly developed device along with the conceptual technology. By open-loop or close-loop algorithm individually controlling the amplitude and delay of the radiating element and via the interaction of sound waves, the phased array can excite the stable single-mode sound field or multi-mode sound field with some certain directivity and achieve the remote target detection with the higher energy efficient. In the shallow water acoustic aspects, by the vertical phased linear array, the theoretical, experimental and applied researches are just getting started. In his thesis [1], Buck used the phased array to excite the single mode by the close-loop feedback control algorithms. In [2], Gingras used the single-mode sound field excited by the open-loop control to research the attenuation and backscattering in shallow water. Recently, Russian and Chinese scholars took part in the research of the single-mode excitation. In [3], Russian scholars used a vertical linear array to excite the single-mode. Combined with the space filter, simulations studied on the lower number normal mode in [4]. The Chinese scholars solved the broadband matching [5] and poor consistency issues of phased array, finally developed a vertical array system to excite the single-mode in shallow water. Now, they formed a series of sound field control algorithms in near-filed. However, in respect of employing the phased array to remote detection, it remained to be studied of how to choose the weight vector, and yet it was missing evaluation criteria.

In this paper, we introduce the Cramer-Rao bound of time-delay and Doppler frequency-shift estimation to assess the far-field sound field control effects. Using the information theory and classical estimation theory, the variance lower bound of parameter estimation was derived. Similar
work had been done in [6-9]. In [6], Based on the radar-target model, Aleksandar researched on the parameter estimations of an active array in pure mathematical derivation. In [7], Swingler used CRB to evaluate the direction of arrival of two adjacent targets. In [8], the maximum mutual information was used to design the non-uniform array. In [9], Saurav researched on optimal array design by maximizing Fisher information for bearing estimation. In this paper, firstly, the sonar-target model excited by the phased array is derived. Secondly, employing the information matrix, the relationship between CRBs and weight vector are provided. At last, under the conditions of same energy input, the far-field sound field control effects emitted by several weight vectors of phased array are acquired.

2. Derivations

From the adiabatic normal mode theory, the sonar-target model emitted by the phased array is derived. After that, combining the information theory, the relationship between the CRBs of parameter (time-delay, Doppler frequency-shift) estimations and the weight vector are set up.

2.1 Sonar-target model emitted by the phased array

In the shallow waveguide, combining the normal mode theory, the sound field emitted by the phased vertical linear array is represented as follow

\[ p(r,z) = i\pi \sum_{n=1}^{N} \sum_{m=1}^{M} \psi_m(z) \psi_m(z) H_0^{(1)}(k_mr) w_n \]  

(1)

For \( m = 1,2,\cdots, M \), \( n = 1,2,\cdots, N \), \( M \) the number of normal mode, \( N \) the number of phased array elements, where \( (z,r) \) the location of the receiver; \( \psi_m \) the eigenfunctions of normal mode; \( H_0^{(1)}(\cdot) \) the Hankel function; \( k_m \) the eigenvalues.

If the signal \( S(\omega) \) in frequency domain form is used by the phased array, the receiver gets the signal in frequency domain as follow

\[ p(r,z,\omega) = i\pi \sum_{n=1}^{N} \sum_{m=1}^{M} \psi_m(z) \psi_m(z) H_0^{(1)}(k_mr) w_n S(\omega) \]  

(2)

Using the asymptotic approximation to the Hankel function and the normal mode decomposition technique, the sound field in frequency domain of receiver is represented as follow

\[ p(r,z,\omega) = \sum_{m=1}^{M} \frac{A_m}{\sqrt{8\pi}} \cdot \exp(ik_mr - i\pi/4) \cdot S(\omega) \]  

(3)

for \( W = [w_1, w_2, \cdots, w_N]^T \) the phased array weight vector, \( T \) the transpose operator; \( W_m \), \( m = 1,2,\cdots, M \) the normal mode component, where \( A_m = \frac{\psi_m(z) \cdot \psi_m^T W_m}{\sqrt{k_m}} \cdot \exp(-\alpha_m) \), the complex amplitude of normal mode, \( \alpha_m \) the attenuation coefficient of normal mode.

Using the inverse Fourier transform, the receiver sound field in time domain is acquired as follow

\[ p(r,z,t) = \int p(r,z,\omega) \cdot e^{-i\omega t} \, d\omega \]  

(4)

Ignoring the correlation between the eigen-functions and frequency range in narrowband and introducing the group delay \( \tau_m = \frac{R}{v_m} \), \( m = 1,2,\cdots, M \), where \( R \) is the range and \( v_m \) is the group velocities of normal mode.

\[ p(r,z,t) = \sum_{m=1}^{M} \frac{1}{\sqrt{8\pi}} \int A_m \exp(ik_mr - i\pi/4) \cdot S(\omega) \cdot e^{-i\omega(t-\tau_m)} \, d\omega \]  

(5)
In the frequency bandwidth of the narrow-band signal \( S(\omega) \), \( A_m, k_m, \tau_m, \alpha_m \) are almost constant. From Eq.(5), the receiving sound field representation is the weighted sum of normal mode. It's shown as follow

\[
p(r, z_i, t) = \sum_{m=1}^{M} A_m \exp(ik_m r - i\pi / 4) \cdot \int S(\omega) \cdot e^{-i\omega(t-\tau_m)} \, d\omega
\]

From Eq.(6), it's simplified as follow

\[
p(r, z_i, t) = \sum_{m=1}^{M} B_m \cdot s(t - \tau_m), \quad B_m = \frac{A_m \exp(ik_m r - i\pi / 4)}{\sqrt{8\pi}}
\]

As a result of compensating the group delay of normal mode, eq.(7) is written as

\[
p(r, z_i, t) = x \cdot s(t - \tau)
\]

where \( x = Q(r, z_i)W = \sum_{m=1}^{M} B_m \cdot D(t - \tau + \tau_m) \) and \( D(t - \tau + \tau_m) \) the operator for delay differential compensation. From Eq.(8) extracting the complex envelope signal, and then adding the complex Gaussian white noise and taking the Doppler frequency shift into consideration. Finally we get the sonar-target model in continuous time domain as

\[
p(r, z_i, t) = x \cdot e^{i\Omega_D \cdot t} s(t - \tau) + e(t)
\]

where \( \Omega \) the Doppler frequency shift value, \( e(t) \) the Gaussian white noise.

Generally, we get the far-field sound field representation. Base on sonar-target model, the relationship between the parameter estimation CRBs and the weight vector of phased array will be set up later. In this paper, by CRBs, the phased array emitting different weight vectors are discussed. the results of comparison with the conventional sonar system are provided in simulation. Even though the optimal weight vector is interesting, this paper doesn't discussed further and leave to discuss in the subsequent articles.

### 2.2 The CRBs of time-delay and Doppler frequency shift estimation

Combining the classical estimation theory, the parameter estimation variance lower bounds are used to evaluate the quality of algorithm as well as the far-field sound field control effects. Though many theoretical lower bounds exist, this paper chooses the Cramer-Rao bounds as the evaluation standard of far field control effect.

From Eq.(9), after converting to base-band and sampling, the discrete sonar-target model is written as

\[
y[n] = x \cdot e^{i\omega_D n} s[n - n_r] + e[n]
\]

where \( \omega_D = \Omega_D \cdot \Delta t \); \( n_r = \tau / \Delta t \); \( t = n \cdot \Delta t \); \( \Delta t \) the sampling interval.

For the Eq.(10) to be valid, several assumptions need to be satisfied. Firstly, To model the Doppler effect by a frequency shift, the radial component of the target's velocity needs to be much smaller than the propagation speed \( v / c << 1 \). Secondly, the time-bandwidth product \( W_a T \) of the complex envelope \( s(t) \) should be much smaller than \( c / 2v \). Finally, \( n = 1, 2, \ldots, N \) covers the coherent processing interval (CPI).

According to the Eq.(10), to start from the Fisher information matrix, the CRBs of time-delay and Doppler frequency shift estimation are derived. Using the vector representation, the sonar target model is rewritten as

\[
y = \mu(\gamma) + e
\]

where \( \mu = [\omega_D, n_r)^T \), \( \gamma = [\text{Re}\{x\}, \text{Imag}\{x\}, \text{ve}^T] \), \( \mu[n, \gamma] = x \cdot \exp(j\omega_D n) s[n - n_r] \),

\[
\]

\[
\mu(\gamma) = [\mu[1, \gamma]^T, \mu[2, \gamma]^T \ldots \mu[N, \gamma]^T]^T, \quad \varphi(\mathbf{v}) = \mu(\gamma) / x.
\]
**is the additive noise with statistical properties as** $E(e) = 0$, $E(ee^*) = C$, thus, $y$ has the statistical properties as $E(y) = \mu(y)$, $C_y = E(yy^*)$, where * the conjugate operator; $^H$ the conjugate transpose operator. According to classical estimation theory of complex data, the complex Gaussian probability density function is written as follow

$$p(y; \nu) = \frac{1}{\pi^N \det(C_y(\nu))} \exp[-(y - \mu(y))^H C_y^{-1}(\nu)(y - \mu(y))]$$

\text{ Eq. (12) }

\text{ \vdash \quad } C_y(\nu) = C \text{, the Fisher information matrix is shown as follow}

$$[I(\gamma)]_{ij} = 2 \text{Re} \left[ \frac{\partial \mu(y)}{\partial \gamma_i} C^{-1} \frac{\partial \mu(y)}{\partial \gamma_j} \right]$$

\text{ Eq. (13) }

In the Eq. (13), the every partial derivatives have the following expressions

$$\frac{\partial \mu(y)}{\partial \text{Re}(x)} = [\exp(j\omega_D)s[1-n_r], \exp(j2\omega_D)s[2-n_r], \cdots \exp(jN\omega_D)s[N-n_r]]^T,$$

$$\frac{\partial \mu(y)}{\partial \text{Imag}(x)} = [j \exp(j\omega_D)s[1-n_r], j \exp(j2\omega_D)s[2-n_r], \cdots j \exp(jN\omega_D)s[N-n_r]]^T,$$

$$\frac{\partial \mu(y)}{\partial \omega_D} = [jx \cdot \exp(j\omega_D)s[1-n_r], 2 jx \cdot \exp(j2\omega_D)s[2-n_r], \cdots N jx \cdot \exp(jN\omega_D)s[N-n_r]]^T,$$

$$\frac{\partial \mu(y)}{\partial n_r} = [x \cdot \exp(j\omega_D)d[1-n_r], x \cdot \exp(j2\omega_D)d[2-n_r], \cdots x \cdot \exp(jN\omega_D)d[N-n_r]]^T,$$

\text{ where } $d[n-n_r] = -\frac{\partial \delta[n-n_r]}{n_r}$.

\text{ to define } $A = \begin{bmatrix} [I(\gamma)]_{11} \\ [I(\gamma)]_{12} \\ [I(\gamma)]_{21} \\ [I(\gamma)]_{22} \end{bmatrix}$, $B = \begin{bmatrix} [I(\gamma)]_{33} \\ [I(\gamma)]_{34} \\ [I(\gamma)]_{43} \\ [I(\gamma)]_{44} \end{bmatrix}$, $D = \begin{bmatrix} [I(\gamma)]_{33} \\ [I(\gamma)]_{34} \\ [I(\gamma)]_{43} \\ [I(\gamma)]_{44} \end{bmatrix}$

so, the Information matrix turns into block matrix as follow

$$I(\gamma) = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

\text{ Eq. (14) }

According to the relation between information matrix and CRBs, we get the CRBs as follow

$$\text{ CRB}(\gamma) = \begin{bmatrix} A^{-1} + A^{-1}B(D - B^TA^{-1}B)^{-1}B^TA^{-1} & -A^{-1}B(D - B^TA^{-1}B)^{-1} \\ -D(B - B^TA^{-1}B)^{-1}B^TA^{-1} & (D - B^TA^{-1}B)^{-1} \end{bmatrix}$$

\text{ Eq. (15) }

Here, we only take the CRBs of time-delay and Doppler frequency shift estimation into consideration, thus

$$\text{ CRB}(\nu) = (D - B^TA^{-1}B)^{-1} = \frac{\sigma^2}{4|x|^2} \begin{bmatrix} \lambda & \text{Imag}(\xi) \\ \text{Imag}(\xi) & \rho \end{bmatrix}^{-1}$$

\text{ Eq. (16) }

where, $\varepsilon = \sum_{n=1}^{N}[s[n-n_r]], \lambda = \frac{1}{\varepsilon} \sum_{n=1}^{N}[s[n-n_r]]^2 - \frac{1}{\varepsilon} \sum_{n=1}^{N}[s[n-n_r]]^2$, $\xi = \sum_{n=1}^{N}s[n-n_r]d[n-n_r]$, $\rho = \frac{1}{\varepsilon} \sum_{n=1}^{N}[s[n-n_r]]^2 d[n-n_r]$, $\text{SNR} = |x|^2 \cdot \varepsilon / (\sigma^2 / 2)$ is the signal noise ratio.
In general, from Eq.(16), $CRB(\psi)$ can be divided into two parts: (1) $\sigma^2 / 4|\lambda|^{2}$ is the far-field sound field intensity emitted by phased array with weight vector on the contribution of parameter estimation CRBs. It represents the forward channel characteristics. From the formula, CRBs are the inversely proportional to the incident sound field intensity. (2) is the relevant parts of the signal, we don't discuss in this paper.

From the above derivation, increasing the incident sound field intensity, the CRBs of time delay and Doppler frequency shift estimation decrease. However, to the active sonar system, the maximal source level limits the incident sound field intensity. The phased array sonar system can solve the problem, through the high efficient energy conversion; it achieves the optimal far-field sound control effects. In this paper, we don't give how to solve the optimal weight vectors theoretically. It will be discussed in the subsequent articles. In simulation, we will give the optimal weight vector's far-field control effect.

3. Simulations

In this section, the evaluation standard proposed in this paper for the far-field sound field control effect is studied by numerical simulations in a range independent waveguide environment (Pekeris model), as shown in Fig.1. The depth of the waveguide is 32.5m and the acoustic parameters of the water layer and bottom are presented in Fig.1. The phased array consists of 21 elements spanning the water volume from 1.5m to 31.5m with 1.5m inter-element spacing. The vertical linear array consists of 32 elements spanning the water volume from 1m to 32m with the 1m inter-element distance. The central frequencies of signal for simulation are 600Hz, 900Hz respectively. The acoustic field and the normal modes in the waveguide are calculated by KRAKEN normal mode codes. The normal mode numbers of 600Hz and 900Hz are 17 and 25 respectively. The ambient noise is Gaussian white noise. 32-element VLA is used to detect far-field sound field control effects of the phased array.

In simulation, firstly, the eigen-functions $\psi_m$ and eigen-values $k_m$ are calculated by KRAKEN program. For the inversely proportional relationships between CRBs and the incident acoustic field intensity, at the same noise level, the acoustic field intensity is employed to show the sound field control effects directly. Under the conditions of same energy input($\mathbf{W}^H \mathbf{W} = 1$), the following
weight vectors are considered: 
\[
W_0 = \frac{1}{\sqrt{N}} \cdot 1_N, \quad \text{all elements emitted at the same values}
\]
\[
W_1 = \frac{1}{\sqrt{N}} \cdot 1_N, \quad \text{the } N \times 1 \text{ full 1 matrix;}
\]
\[
W_i = \frac{\psi_i(z)}{\sqrt{\psi_i^2(z)\psi_i(z)}}, \quad i = 1, 2, 3, \text{ No.1-3 normal mode.}
\]
\[
W_i, \quad i = 4, 5, 6, \text{ single element emitting at the different (9m, 18m, 27m) depths, simulating the}
\]
conventional sonar system.
\[
W_{opt} \quad \text{the optimal weight vector related to three depths (9m, 18m, 27m).}
\]

All of weight vector emitted by the phased array have the same energy input \(W^H W = 1\). The receiving ranges are 10km, 20km, 30km respectively, and sound field curve is normalized by the maximum value at corresponding range. The results are shown in Fig.2-Fig.7. The difference between \(W_0\), \(W_i\) and \(W_{opt}\) is listed in table 1.

![Fig.2](image1.png)  
**Fig.2** the normalized sound intensity of 600Hz at 10km  
![Fig.3](image2.png)  
**Fig.3** the normalized sound intensity of 900Hz at 10km  

![Fig.4](image3.png)  
**Fig.4** the normalized sound intensity of 600Hz at 20km  
![Fig.5](image4.png)  
**Fig.5** the normalized sound intensity of 900Hz at 20km.
Table 1: The difference of the incident sound intensity

<table>
<thead>
<tr>
<th>Range (km)</th>
<th>Depth (m)</th>
<th>9 m (dB)</th>
<th>18 m (dB)</th>
<th>27 m (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W₀-W₀pt</td>
<td>W₁-W₀pt</td>
<td>W₀-W₀pt</td>
<td>W₁-W₀pt</td>
</tr>
<tr>
<td>600 Hz</td>
<td>10 km</td>
<td>-3.06</td>
<td>-2.91</td>
<td>-1.66</td>
</tr>
<tr>
<td></td>
<td>20 km</td>
<td>-2.28</td>
<td>-1.50</td>
<td>-0.55</td>
</tr>
<tr>
<td></td>
<td>30 km</td>
<td>-1.51</td>
<td>-0.72</td>
<td>-0.51</td>
</tr>
<tr>
<td>900 Hz</td>
<td>10 km</td>
<td>-3.93</td>
<td>-4.15</td>
<td>-3.90</td>
</tr>
<tr>
<td></td>
<td>20 km</td>
<td>-2.75</td>
<td>-2.89</td>
<td>-2.35</td>
</tr>
<tr>
<td></td>
<td>30 km</td>
<td>-2.22</td>
<td>-2.06</td>
<td>-1.57</td>
</tr>
</tbody>
</table>

From the simulating results, it's shown that the conventional sonar system is the inefficient sound source of remote detection. The phased array emitting the weight vectors \( W_i, i = 4,5,6 \) lead to very weak incident sound intensities. Thus, from the relationship contain in the time-delay and Doppler frequency shift estimation CRBs, it's able to evaluate the far-field sound control effects. Under the condition of the optimal weight vector unknown, probably, the some number of normal mode is the suboptimal according to the environmental parameters. If the environment is unknown, in this paper, the \( W_0 \) is suggested to use for the remote detection weight vector emitted by the phased array.

4. Conclusions

Based on the normal mode theory, the sonar-target model of phased array emitting the different weight vectors is derived and the relationships between parameters (time-delay and Doppler frequency shift) estimation Cramer-Rao bounds and the weight vectors are set up theoretically. Simulations and discussions employing the representation for the evaluation of the far-field sound field control effects; it’s shown that the phased array as the remote detection source is more efficient than the conventional sonar system under the condition of same energy input. The phased array emitting the optimal weight vector is able to reach both of time-delay and Doppler frequency shift estimation variance lower bounds.

So far, in this paper, we don't discuss how to solve the optimal weight vector by the representation of CRBs. After that, it will be discussed in the subsequent articles. In the unknown environment, we suggest that the phased array emitting the weight vector of \( W_0 \) is supposed to using for the remote detection.
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REFERENCES