APPLICATION OF NODAL DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR 2D NONLINEAR ELASTIC WAVE PROPAGATION

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In order to solve the elastic wave equation in heterogeneous media with arbitrary high order accuracy in space on unstructured meshes, a nodal Discontinuous Galerkin Finite Element Method (DG-FEM) is presented, which combines the geometrical flexibility of the Finite Element Method and strongly nonlinear wave simulation capability of the Finite Volume Method. The equations of nonlinear elastodynamics have been written in a conservative form in order to facilitate the numerical implementation and introduce different kinds of elastic nonlinearities, such as the classical nonlinearities and non-classical hysteretic nonlinearities. In the calculation of DG-FEM scheme, different kinds of boundary conditions and numerical fluxes have been discussed. The numerical simulations of linear elastic wave propagation and plane wave nonlinear propagation demonstrated the developed DG-FEM scheme has an excellent precision and performance in numerical application.

1. Introduction

In recent years, a strong interest for Nondestructive Testing (NDT) methods based on nonlinear elastic effects in solid has grown, driven by the request from industry for sensitive quantification and localization of micro-structural damage. In order to do that, the fundamental study of nonlinear signatures of the elastic waves in heterogeneous media with complex geometries and given a reliable simulation model of wave propagation are necessary and prerequisite.

Now, for different numerical application and nonlinear study, some different numerical methods have been applied for numerical simulation of nonlinear elastic wave propagation in heterogeneous media. Due to the features of simplicity and robustness, the Finite Difference Time Domain (FDTD) method has been used to deal with nonlinear seismic wave propagation in anisotropic media using the flux-corrected transport technique\textsuperscript{1}. In numerical calculation, the FDTD method ill-suited to solve complex geometries. For the Finite Volume Method (FVM), his most important feature is that it can be used on arbitrary geometries, using structured or unstructured meshes, due to the introduction of an element-based discretization. Zumpano et al.\textsuperscript{2} used Finite Element Method (FEM) to realize the simulation study for the identification and localization of stress corrosion cracking by a nonlinear elastic time reversal acoustic technology. The FEM method can introduce
more degree of freedom on the mesh element than FVM. Due to the efficiency in solving large scale problems the Pseudo-Spectral (PS) method has been applied for nonlinear hysteretic media.\(^4\)

For the simulation of nonlinear elastic wave propagation in complex geometry structures, a high-order numerical method with a high degree of accuracy and geometric flexibility is required. The FEM has the properties of geometric flexibility and high-order accuracy, however, its globally defined basis and test functions destroy the locality of the scheme and introduce potential problems of stability for wave-dominated problems. An intelligent combination of the finite element and the finite volume methods, utilizing a space of basis and test functions that mimics the finite element method but satisfying the equation in a sense closer to the finite volume method, appears to offer many of the desired properties. This combination is exactly what leads to the Discontinuous Galerkin Finite Element Method (DG-FEM) which has been proposed firstly as a way of solving the steady state neutron transport equation.\(^4\)

There are two kinds of DG-FEM scheme: modal DG-FEM and nodal DG-FEM, we chose to work with the nodal DG-FEM scheme first introduced by Hesthaven and Warburton\(^5\) for electrodynamic simulations. Moreover, a numerical scheme based on this method has been developed since then for a large number of fields as left-handed medium\(^6\), water-wave and free surface flow\(^7\), piezoelectric transducer\(^8\), and nonlinear elastic wave\(^9\).

2. The nodal discontinuous Galerkin finite element method scheme

2.1 Basic nonlinear elastodynamics equations

For nonlinear elastodynamic solid medium, with classical and non-classical nonlinearities, it is more judicious to discretize the fundamental elastodynamic equations expressed in conservation form. The considered equation of motion can be written, with Einstein’s convention of summation:

\[
P_0 \frac{\partial v_i}{\partial t} = \frac{\partial P_{ij}}{\partial a_j},
\]

where \(v_i\) are the components of the particle velocity vector, \(a_j\) are the components of the Lagrangian position vector, \(\rho_0\) is the density, \(P_{ij}\) are the components of the Piola-Kirchoff tensor, and \(t\) is the time. These equations are completed by the constitutive relation:

\[
P_j = \rho_0 \frac{\partial W}{\partial F_{ij}},
\]

where \(W\) is the elastic energy density which depends on the considered nonlinear (or linear) model of elasticity, and \(F\) is the deformation gradient:

\[
F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial a_j},
\]

where \(\delta_{ij}\) is the Dirac delta function, and \(u_i\) are the component of the displacement vector. For example, in the case of an anisotropic linear elastic solid the constitutive equations are given by the Hooke’s law:

\[
P_j = \tau_{ij} = C_{ijkl} \frac{\partial u_k}{\partial a_l},
\]

where \(C_{ijkl}\) are the elastic constants. Finally, the system is closed by the link between particle velocity and deformations gradient:

\[
\frac{\partial F_{ijkl}}{\partial t} = \frac{\partial v_i}{\partial a_j}.
\]

To resume, the system to be solved, in order to simulate propagation of elastic waves in nonlinear elastic solids in 3D, is written in the following conservation form:

\[
\frac{\partial Q(t,x)}{\partial t} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}, \quad x = [x, y, z] \in \Omega \in \mathbb{R}^3,
\]
where \( \Omega \) is the global physical domain with boundary \( \partial \Omega \) and \( \mathbf{Q} \) is the variables vector, \( \mathbf{F}_x \), \( \mathbf{F}_y \), and \( \mathbf{F}_z \) are fluxes in the \( x \), \( y \) and \( z \) direction respectively. Equation (6) is the complemented by a constitutive equation, e.g., Eq. (4) for the linear approximation or Eq. (23)-(24) in the nonlinear case, which links the deformation gradient and the stress tensor. The reasons of this conservation form has been adopted are that: Firstly, it enables a direct extension to the simulation of linear elastic wave propagation or nonlinear elastic wave in stressed material by allowing a non-symmetric stress tensor; Secondly, this form allows to easily introduce different kinds of nonlinearity models, e.g., five-constant nonlinearity or hysteretic nonlinearity. For the system equations of this conservation form in 2D, the vector \( \mathbf{F}_z \) and the variables which have subscript 3 in vectors \( \mathbf{Q} \), \( \mathbf{F}_x \) and \( \mathbf{F}_y \) will disappeared, and at the same time the new \( \mathbf{F}_x \) and \( \mathbf{F}_y \) will be replaced by the new notation vectors \( \mathbf{F} \) and \( \mathbf{G} \). The 2D form will be used in the following as presented in reference [9].

2.2 General formulation of discontinuous Galerkin scheme in 2D

In the discontinuous scheme, the global domain \( \Omega \) is divided into \( K \) non-overlapping elements \( D^k \):

\[
\Omega = \bigcup_{k=1}^{K} D^k .
\]

In the \( k \)-th element, the numerical solution \( \mathbf{Q}_h^k \) of Eq. (8) is approximated through an interpolation

\[
\mathbf{Q}(t, \mathbf{x}) \approx \mathbf{Q}_h^k (t, \mathbf{x}) = \sum_{i=1}^{N_k} \mathbf{Q}_h^k (t, \mathbf{x}_i) \psi_i (\mathbf{x}) = \sum_{i=1}^{N_k} \mathbf{Q}_h^k (t, \mathbf{x}_i) \mathbf{f}_i (\mathbf{x}) = \sum_{i=1}^{N_k} \mathbf{Q}_h^k (t, \mathbf{x}_i) \mathbf{f}_i (\mathbf{x}) .
\]

In the first formulation, known as the modal form, \( \psi_i (\mathbf{x}) \) is a local two-dimensional polynomial basis of order \( N \). In the alternative form, known as the nodal representation, \( \mathbf{f}_i (\mathbf{x}) \) are two-dimensional Lagrange interpolation polynomials based on the grid points \( \mathbf{x}_i \). The connection between these two forms is done through the definition of the expansion coefficients \( \mathbf{Q}_h^k \). \( N_p \) is the number of interpolation grid points in each element, which is equivalent to the number of expansion terms. An interpolation is obtained by connecting these grid points to a set of basis functions.

Multiplying Eq. (6) in 2D form by a test function, the same as the basis function in our case of Galerkin Method, and integrating on each element \( D^k \) yields

\[
\int_{D^k} \frac{\partial \mathbf{Q}_h^k \mathbf{f}_i (\mathbf{x})}{\partial t} d\mathbf{x} = \int_{D^k} \left( \frac{\partial \mathbf{F}_h^k}{\partial \mathbf{x}} + \frac{\partial \mathbf{G}_h^k}{\partial \mathbf{y}} \mathbf{f}_i (\mathbf{x}) \right) d\mathbf{x} .
\]

According to the Green theorem, the first term of Eq. (10) in the right hand can be written as a curl integral on the edges \( \partial D^k \) of element \( D^k \), and the following equation is obtained

\[
\int_{\partial D^k} \mathbf{G}_h^k \times \mathbf{f}_i (\mathbf{x}) d\mathbf{r} = \int_{\partial D^k} \mathbf{f}_i (\mathbf{x}) \times \mathbf{G}_h^k d\mathbf{r} .
\]
\[
\int_{\partial D^k} \frac{\partial Q_k^i}{\partial t} I_k^i(x) dx = \int_{\partial D^k} \left( n_i^k \mathbf{F}_h^k + n_i^k \mathbf{G}_h^k \right) I_k^i(x) dx - \int_{\partial D^k} \left( \frac{\partial F_k^i}{\partial x}(x) + \frac{\partial G_k^i}{\partial y}(x) \right) \mathbf{G}_h^k d\mathbf{x},
\]

where \( \partial D^k \) is the edge of the \( D^k \), and \( \mathbf{n}_k = [n_i^k \ n_j^k] \) is the normalized outward pointing normal vector. Since \( Q_h \) may be discontinuous at an element boundary, we replace the flux \( (n_i^k \mathbf{F}_h^k + n_i^k \mathbf{G}_h^k) = \mathbf{f}^k \) by a numerical flux \( (n_i^k \mathbf{F}_h^k + n_i^k \mathbf{G}_h^k)^* = \mathbf{f}^k^* \):

\[
\int_{\partial D^k} \frac{\partial Q_k^i}{\partial t} I_k^i(x) dx = \int_{\partial D^k} \left( (n_i^k \mathbf{F}_h^k + n_i^k \mathbf{G}_h^k)^* - (n_i^k \mathbf{F}_h^k + n_i^k \mathbf{G}_h^k) \right) I_k^i(x) dx
+ \int_{\partial D^k} \left( \frac{\partial F_k^i}{\partial x} + \frac{\partial G_k^i}{\partial y} \right) \mathbf{G}_h^k d\mathbf{x}.
\]

Equation (12) is the strong formulation of the nodal discontinuous Galerkin method in two spatial dimensions, which allow a space of non-smooth test functions. We use the strong form in the following.

### 2.3 Defining discontinuous Galerkin operators on triangular elements

As in the preceding presentation, the global domain \( \Omega \) is divided into \( K \) non-overlapping triangular elements \( D^k \). The number of interpolation points \( N_p \) for each triangular element, has the following relation with the polynomial order \( N \)

\[
N_p = \frac{(N+1)(N+2)}{2},
\]

The grid points, which the Lagrange interpolation is based on, are a set of local grid points belonging to element \( k \). They define the vectors \( x_k^s = [x_k^1, x_k^2, \ldots , x_k^{N_p}] \) and \( y_k^s = [y_k^1, y_k^2, \ldots , y_k^{N_p}] \). Here, considering the case where we interpolate with the same number of grid points \( N_p \), in all the elements.

In order to connecting the general straight-sided triangle with the standard straight-angle triangle, as sketched in Fig. 1

**Figure** we introduce a mapping, \( \Psi \), defined as

\[
I = \{ r = (r, s) \mid (r, s) \geq -1; r + s \leq 0 \},
\]

**Figure 1.** The mapping between the reference straight-angle triangle element \( I \) and a general triangular shaped element.

where \( r \) and \( s \) are the coordinates in a reference standard triangle. In the reference triangle \( I \), it’s very important how to find exactly \( N_p \) points for interpolation. Here, the one dimensional Legendre-Guass-Lobatto points, defined by the vectors \( r = [r_1, r_2, \ldots, r_{N_p}]^T \) and \( s = [s_1, s_2, \ldots, s_{N_p}]^T \), have been used.

Through the mapping, we are back in the position where we can focus on the development of polynomials and operators defined on \( I \). To obtain a spectral scheme, we need to define an orthogonal set of basis functions on the reference triangle \( I \). One kind of basis functions is

\[
\psi_m(r, s) = \sqrt{2} P_{i}^{(n_k)}(a) P_{j}^{(2i+1, 0)}(b)(1-b)^{j}, \text{ for } (i, j) \geq 0 \text{ and } i + j \leq N,
\]
with \( a = 2(1 + r)/(1 - s) - 1, b = s \) and \( m = i + (N + 1)j + 1 - j/2(j - 1) \). \( P_n^{(\alpha, \beta)}(x) \) is the \( n \)-th order Jacobi polynomial. If \( \alpha = \beta = 0 \), then it becomes the Legendre polynomial.

In the reference triangle, by interpolating, the transformation between modal and nodal form can be achieved:

\[
\sum_{j=1}^{N_f} \hat{Q}_{j}\psi_j(r, s) \approx \sum_{j=1}^{N_f} \sum_{i=1}^{N_e} \hat{Q}_{i, j}\psi_j(r, s) = \sum_{i=1}^{N_e} \hat{Q}_{i, j}\psi_j(r, s).
\]

From this equation and by interpolating the basis function, the two following relations can be obtained in the matrix form: \( Q = V^T \) and \( \psi(r, s) = V^T \hat{\psi}(r, s)\). Here, we have defined the vectors \( r = [r_1, r_2, ..., r_{N_e}] \) and \( s = [s_1, s_2, ..., s_{N_e}] \), and introduce the Vandermonde matrix \( V = \psi_j(r, s) \). \( V^T \) is the transpose of \( V \).

Now, we do not have the direct expression of the derivation of the Lagrange interpolation \( I \) in nodal representation yet, we can calculate the derivatives in modal space and transform the derivatives back to nodal space, because the derivatives of the basis function \( \psi \) can be obtained directly. On the reference element \( I \), the differentiation operator can be computed from the following relations

\[
D_r = \left[ \begin{array}{c} \frac{\partial \hat{Q}}{\partial r} \\ \frac{\partial \hat{Q}}{\partial s} \end{array} \right] = (V^T)^{-1} \hat{\psi}(r, s) = (V^T)^{-1} \left( \begin{array}{c} \frac{\partial \hat{Q}}{\partial r} \\ \frac{\partial \hat{Q}}{\partial s} \end{array} \right),
\]

\[
D_s = \left[ \begin{array}{c} \frac{\partial \hat{Q}}{\partial s} \\ \frac{\partial \hat{Q}}{\partial r} \end{array} \right] = (V^T)^{-1} \hat{\psi}(r, s) = (V^T)^{-1} \left( \begin{array}{c} \frac{\partial \hat{Q}}{\partial s} \\ \frac{\partial \hat{Q}}{\partial r} \end{array} \right).
\]

Using the chain rule, the differentiation matrix is then obtained directly

\[
\frac{\partial}{\partial r} = \frac{\partial}{\partial x} D_r + \frac{\partial}{\partial y} D_s, \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial x} D_s + \frac{\partial}{\partial y} D_r.
\]

By using the differentiation matrix Eq. (19), the calculation of volume term will be obtained as follows

\[
\frac{\partial F^k}{\partial x} + \frac{\partial G^k}{\partial y} = r_s D_s F^k + s_s D_s G^k + r_r D_r F^k + s_r D_r G^k.
\]

2.4 Numerical flux and boundary condition in the discontinuous Galerkin method

In the discontinuous method, the numerical fluxes are typically functions of the information about the both: \( f^k(Q_{b_h}, Q_{h_b}) \). There are many different numerical fluxes that have been suggested in the literature, such as, central flux, Godunov flux and Lax-Freidrich flux. Here, we consider the Lax-Freidrich flux, which has the following formulation:

\[
 f^k_{L^k}(Q_{b_h}, Q_{h_b}) = \frac{f_i(Q_{b_h}^k) + f_i(Q_{h_b}^k)}{2} + \frac{C}{2} \hat{n}^k \cdot (Q_{b_h}^k - Q_{h_b}^k),
\]

\[
= n_s(F(Q_{b_h}^k) + F(Q_{h_b}^k)) + n_s(G(Q_{b_h}^k) + G(Q_{h_b}^k)) + \frac{C}{2} \hat{n}^k \cdot (Q_{b_h}^k - Q_{h_b}^k),
\]

\[
= n_s(F(Q_{b_h}^k) + F(Q_{h_b}^k)) + n_s(G(Q_{b_h}^k) + G(Q_{h_b}^k)) + \frac{C}{2} \hat{n}^k \cdot (Q_{b_h}^k - Q_{h_b}^k),
\]

\[
= n_s(F(Q_{b_h}^k) + F(Q_{h_b}^k)) + n_s(G(Q_{b_h}^k) + G(Q_{h_b}^k)) + \frac{C}{2} \hat{n}^k \cdot (Q_{b_h}^k - Q_{h_b}^k),
\]
This flux ensures a monotone solution and can therefore be used in nonlinear hyperbolic problems.

Among the large variety of physically meaningful boundary conditions exists for an elastic medium, the three most important kinds of boundaries are: open boundaries, stress free surface boundaries and fixed surface boundaries. Here, stress free surface boundaries will be considered. On the free surface of an elastic medium, the normal stress and the shear stresses with respect to the boundary are determined by physical constraints. At the outside of the elastic medium, there are no external forces that retract the particles into their original position. Therefore, the normal stress and the shear stress values at the free surface have to be zero: $P_{\xi\xi} = 0$ and $P_{\xi\eta} = 0$. Here, $\xi$ and $\eta$ indicates the normal and tangential directions, respectively.

3. Numerical validation of DG-FEM for elastic wave propagation

In this section, two simulations will be presented to demonstrate the performance of the proposed DG-FEM scheme. Applications of DG-FEM will be made for both linear and nonlinear elastic waves.

3.1 Linear isotropic simulation of Lamb’s problem

Here, for the application of DG-FEM scheme to isotropic medium, we present a classical test case which uses a vertical force in a homogeneous elastic half-space with a free surface. This test is called Lamb’s Problem. The solution of Lamb’s Problem for a plane surface can be computed analytically and can hence be used for comparison with the DG-FEM framework results. We use the FORTRAN code EX2DDIR of Berg to compute the exact solution of the seismic 2D response from a vertical directional point source in an elastic half space.

The numerical model is a rectangular zone (4000 × 2500 m) with origin (0, 0) at the left upper corner. The vertical directional point source is located at the centre position of the underside stress free surface (2000, -2500) and one receiver is located at (2800, -2500). For the homogeneous elastic medium, the parameters of simulation are: the mass density $\rho_0 = 2200$ kg/m$^3$, the velocities of $P$-wave and $S$-wave $c_p = 3200$ m/s and $c_s = 1847.5$ m/s, respectively. The source time function that specifies the temporal variation of the point source is a Ricker wavelet given by the following form:

$$s(t) = (0.5 + a_1(t - t_D)^2) \cdot e^{i(\omega c_2 - t_D)^2}, \quad (23)$$

where $t_D = 0.16$ s is the source delay time and $a_1 = - (\pi f_c)^2$ is the constant determining the amplitude, and the central frequency is $f_c = 7.25$Hz.

In this simulation, the time $T = 1.0$ s and the whole physical domain has been meshed with 4000 triangular elements. The Lax-Friedrich numerical flux and stress free surface boundary condition have been used. The snapshot of the velocity component of the wave field at $t = 0.8$ s presented in Fig. 2, obtained with a seven order DG-FEM scheme, shows the propagation of both the bulk waves and the Rayleigh wave. In Fig. 3, the results of numerical simulations, as recorded by the receiver, together with the analytical solution provided by EX2DDIR have been plotted. The analytical and numerical solutions match well for the vertical and horizontal particle velocities.

Figure 2. Amplitude of the velocity $v$ at $t = 0.8$ s within seven order DG-FEM scheme.
3.2 Simulation of nonlinear elastic plane wave propagation

Here, a validation of the nonlinear elastic wave DG-FEM scheme will be presented. As only a few analytical results are available in the case of nonlinear elastic waves, we consider plane wave propagation in the $x$ direction in a nonlinear elastodynamic medium, in which the stress and strain relationships are explicitly given by\(^{10}\):

\[
P_{11} = C_{11}F_{11} + \beta C_{11}F_{11} + 4\gamma C_{11}F_{21},
\]
\[
P_{12} = (2C_{66} + 4\gamma C_{11}F_{11})F_{11},
\]

where $C_{11} = \lambda + 2\mu$, $C_{66} = \mu$ and $C_{12} = \lambda$, the $\lambda$ and $\mu$ are the Lame constants and the nonlinear parameters $\beta$ and $\gamma$ are equal to 5000 and 4000, respectively. A $3000 \times 200$ m computational domain, with its origin $(0, 0)$ at the centre point, is meshed with 360 triangular elements. A source, consisting of compressional and shear forces, is applied along a line located at $x = -1000$ m in order to generate a plane wave:

\[
s_x(t) = A_x \sin(2\pi ft)e^{-(t-\delta t/2T)^2} \delta(x),
\]
\[
s_y(t) = A_y \sin(2\pi ft)e^{-(t-\delta t/2T)^2} \delta(y),
\]

where $A_x = 1000$ and $A_y = 2000$ are the amplitudes of the sources, $f = 1/T = 20$ Hz is the source frequency and $\delta$ is the Dirac function. Three receivers $R_1$, $R_2$, $R_3$ are positioned at the positions $(-400, 0)$, $(200, 0)$ and $(1000, 0)$, respectively. The Lax-Friedrich numerical flux has been used in the 5 order DG-FEM scheme. The total calculation time is 2 s. Symmetric boundary conditions on the upper and bottom borders, and 500 m thickness Nearly Perfectly Matched Layer (NPML) boundary condition on the left and right sides have been used.

A snapshot of the particle velocity component of the wave field at $t = 0.48$ s is plotted in Fig. 4. This figure shows the plane character of the propagating wave-front and the absorption by the NPML of left going wave generated by the source. In Fig 5, the spectra of the horizontal and vertical particle velocities received at the three different receivers are plotted. Since there is no attenuation, the amplitude of the spectral peak at the source frequency remains nearly unchanged with distance, as expected. However, for the harmonic waves (right spectra in Fig. 5), the evolution as a function of distance is more complex than the linear increase predicted for a plane wave in a fluid. This simulation validates our nonlinear implementation.

Figure 3. Comparison of the analytical reference solution with results of simulation at the receiver within the seven order DG-FEM scheme. The right hand figure displays the horizontal particle velocity and the left hand figure displays the vertical particle velocity.

Figure 4. Velocity amplitude of the plane wave at $t = 0.48$ s obtained with five order DG-FEM scheme.
Figure 5. Spectra of the horizontal $V_x$ and vertical $V_y$ particle velocities of the numerical solutions of the propagation of an elastic plane wave at distances of 3 (dotted line), 6 (dashed line) and 10 (solid line) longitudinal wavelengths.

4. Conclusion

A nodal Discontinuous Galerkin Finite Element Method (DG-FEM) scheme, which is an intelligent combination of the FEM and FVM methods, has been presented. The results of simulations for isotropic Lamb’s problem and nonlinear elastic plane wave have authorized a validation of the DG-FEM scheme.

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REFERENCES