MODIFIED RUNGE_KUTTA METHOD FOR SOLVING NONLINEAR VIBRATION OF AXIALLY TRAVELLING STRING SYSTEM

Qun Wu, Enwei Chen*, Yimin Lu, Zhengshi Liu, Xiang Tang
School of Mechanical and Automotive Engineering, Hefei University of Technology, Hefei, China 230009

e-mail:wq6436200@163.com

In this paper, based on the classical Fourth-Order Runge-Kutta method, the modified Fourth-Order Runge-Kutta method is presented for solving nonlinear vibration of axially travelling string system, that is to solve time varying and nonlinear differential equations. The classical Fourth-Order Runge-Kutta method can only be used to solve first-order linear differential equations. Its main idea is to calculate the value of the function for the state equation four times and then take a linear combination of these values, which increases the order of the truncation error for the numerical method, so as to improve the accuracy of the method. The modified Fourth-Order Runge-Kutta method is a novel method considering the nonlinear term of differential equations, it applies to whatever nonlinear equations that can be transformed into first order matrix differential equation and has no restrict of the order of the original equations. Compared with other methods, it avoids the matrix inversion and has high precision and simplicity. Then a time-varying nonlinear governing equation of an axially travelling string system will be considered, by setting different parameters, the corresponding magnitude of the nonlinear term of the governing equation is different. The modified Fourth-Order Runge-Kutta method and the Newmark-Beta method are used respectively to solve the time-varying nonlinear governing equation of the string model. Good agreements are observed in the results by the two methods, which proved the validity and effectiveness of the modified Fourth-Order Runge-Kutta method and also by setting different parameters indicated that the proposed method is suitable for the differential equations both with strong and weak nonlinear terms.

1. Introduction

In many scientific areas, especially in modeling the axially travelling string system, nonlinear differential equations are difficult to solve exactly. Many researchers have endeavored to find the numerical solutions for the problem. Based on a general weighted residual method, He\textsuperscript{1} proposed a variational method to solve nonlinear problems. Liao\textsuperscript{2} described an optimal homotopy-analysis approach contains at most three convergence-control parameters and demonstrated it is computationally rather efficient. A precise time step integration algorithm and associated matrix exponentiation computations was introduced by zhong\textsuperscript{3,4} and he also pointed out that the precise integration method can be applied both for time history integration and two-point boundary value problems. In Refs.\textsuperscript{5,6}, Zhang constructed an improved precise integration method to solve an augmented Lie type dynamic
system converted by the nonlinear dynamic system. He also proposed a precise integration compounded with Runge–Kutta method as well as a new effective integration method for solving nonlinear stiff invariant dynamical system. While Chen proposed the Newmark-Beta method for solving vibration of an elastic moving string system. In this paper, based on the classical Fourth-Order Runge-Kutta method, the modified Fourth-Order Runge-Kutta method is proposed for solving nonlinear vibration of axially travelling string system and compared with other methods, which is verified suitable for the differential equations both with strong and weak nonlinear terms.

The arrangement of this paper is as follows. Section 2 gives the derivation of the modified Fourth-Order Runge-Kutta method based on the classical Fourth-Order Runge-Kutta method. Section 3 presents an axially travelling string model as example to testify the validity and effectiveness of the modified Fourth-Order Runge-Kutta method for solving nonlinear vibration. Section 4 draws conclusions.

2. Derivation of the modified Fourth-Order Runge-Kutta method

A class of differential equations with nonlinear term derivated from engineering mechanics model after discretization can be written in the following form:

\[ \dot{y} = f(t, y, \phi) . \] (1)

Where \( t \) is a time variable, \( y=[y_1(t), y_2(t), \ldots, y_n(t)]^T \) is an \( n \)-dimensional state vector and \( \phi=\phi(y_1(t), y_2(t), \ldots, y_n(t)) \) is an \( n \)-dimensional vector-valued function as a nonlinear term of the Eq. (1).

For the Eq. (1), we propose a modified Fourth-Order Runge-Kutta method based on the classical Fourth-Order Runge-Kutta method. Assuming the duration of each time step is \( h \), \( t_i=ih \), \( i=1, 2, \ldots \) then the computational formulas over the time interval \([t_i, t_{i+1}]\) are as follow:

\[
\begin{align*}
K_1 &= f(t_i, y(t_i), \phi) \\
K_2 &= f(t_i + \frac{h}{2}, y(t_i) + \frac{h}{2} K_1, \phi_1) \\
K_3 &= f(t_i + \frac{h}{2}, y(t_i) + \frac{h}{2} K_2, \phi_2) \\
K_4 &= f(t_i + h, y(t_i) + h K_3, \phi_3)
\end{align*}
\] (2)

Where we assuming \( K_1=[a_1, a_2, \ldots, a_n]^T, K_2=[b_1, b_2, \ldots, b_n]^T, K_3=[c_1, c_2, \ldots, c_n]^T \), then

\[
\begin{align*}
\phi_1 &= \phi(y_1(t_i) + \frac{h}{2} a_1, y_2(t_i) + \frac{h}{2} a_2, \ldots, y_n(t_i) + \frac{h}{2} a_n) \\
\phi_2 &= \phi(y_1(t_i) + \frac{h}{2} b_1, y_2(t_i) + \frac{h}{2} b_2, \ldots, y_n(t_i) + \frac{h}{2} b_n) \\
\phi_3 &= \phi(y_1(t_i) + h c_1, y_2(t_i) + h c_2, \ldots, y_n(t_i) + h c_n)
\end{align*}
\] (3)

Substitute the Eq. (3) into the Eq. (2), \( K_1, K_2, K_3, K_4 \) can be obtained, so the main computational formula is

\[ y(t_{i+1}) = y(t_i) + \frac{1}{6} h (K_1 + 2K_2 + 2K_3 + K_4). \] (4)
3. Example

Consider the dynamic model of an axially travelling string system proposed by Chen\(^7\), its governing equation is

\[
M(t)\ddot{Q} + C(t)\dot{Q} + K(t)Q + N(t) = 0.
\]

(5)

The model is an axially travelling string with time varying length \(l(t)\) and a speed of \(v(t)\). \(w(x,t)\) is the transverse displacement of the string at time \(t\) and at position \(x\), \(x\) is the axial position of a point in the string and \(t\) is time. \(M(t), C(t), K(t), N(t)\) is derivated by Chen\(^7\), \(Q(t)\) is the transverse displacement of fixed point in the string at time \(t\). The axially travelling string model is discretised with \(n\) elements, each element has the same length, so there are \((n+1)\) nodes, then \(Q=[Q_1(t), Q_2(t), \ldots, Q_{n+1}(t)]^T\). The initial condition is chosen to be the same with Chen\(^7\).

By setting two different parameters shown in Table 1, the number of elements is \(n\), the length of the string is \(l\), the initial amplitude is \(A_0\), the time step is \(\Delta t\), the tension is \(T_0\), the mass per unit length is \(\rho\), the product of young’s modulus \(E\) and the cross-sectional area of string \(A\) is \(EA\), and the number of time steps is \(N\). Using the state space function to solve the linear system response and the Newmark-Beta method to solve the nonlinear system response, the results of some points are shown in Fig. 1(a) and (b).

**Table 1.** parameters of the axially travelling string model.

<table>
<thead>
<tr>
<th>parameters</th>
<th>(n)</th>
<th>(l/m)</th>
<th>(A_0/m^2)</th>
<th>(\Delta t/s)</th>
<th>(T_0/N)</th>
<th>(\rho/(kg/m))</th>
<th>(EA/N)</th>
<th>(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10</td>
<td>3</td>
<td>0.01</td>
<td>0.02</td>
<td>10</td>
<td>1</td>
<td>1000</td>
<td>200</td>
</tr>
<tr>
<td>b</td>
<td>10</td>
<td>3</td>
<td>0.05</td>
<td>0.02</td>
<td>10</td>
<td>1</td>
<td>2.5×10^6</td>
<td>200</td>
</tr>
</tbody>
</table>

![Figure 1](image1.png)

**Figure 1.** Displacement responses of the axially travelling string with constant length.

Key: ——, Displacement responses of linear model using the state space function method,

\(++++\), Displacement responses of nonlinear model using the Newmark-Beta method.

Fig. 1(a) shows good agreement by the two methods, but there exists difference in Fig. 1(b), which indicates by setting different parameters the nonlinear term of the governing equation is different, so the nonlinear term of the governing equation plays significant effect in the response and the nonlinear term for the second parameter set is stronger than the first.
Using the modified Fourth-Order Runge-Kutta method proposed in this paper, the governing equation of the axially travelling string model can be transformed into first order matrix differential equation as follow:

\[
\begin{bmatrix}
\dot{Q} \\
\ddot{Q}
\end{bmatrix} = \begin{bmatrix}
-M^{-1}C & -M^{-1}K \\
I & O
\end{bmatrix} \begin{bmatrix}
\dot{Q} \\
Q
\end{bmatrix} + \begin{bmatrix}
-M^{-1}N \\
O
\end{bmatrix}.
\] (6)

Where setting 

\[y = \begin{bmatrix}
\dot{Q} \\
Q
\end{bmatrix}^T, \quad N_s(t) = \begin{bmatrix}
-M^{-1}N \\
O
\end{bmatrix},\]

then the equation becomes:

\[\dot{y} = f(t, y, N_s(t)).\] (7)

By Matlab simulation, the numerical results obtained from the modified Fourth-Order Runge-Kutta method and the Newmark-Beta method are shown in Fig. 2(a) and (b), good agreements can be observed in the results by these two methods, which testified the validity and effectiveness of the proposed method in this paper. Fig. 2(a) and (b) show the model is set with two different group parameters, but both have the same agreement, while the different parameters means the nonlinear term of the governing equation is different, so it can be concluded that the modified Fourth-Order Runge-Kutta method is suitable for the differential equations both with strong and weak nonlinear terms.

![Figure 2](image.png)

**Figure 2.** Displacement responses of the axially travelling string with constant length.

Key: ———, Displacement responses of nonlinear model using the modified Fourth-Order Runge-Kutta method,

+++, Displacement responses of nonlinear model using the Newmark-Beta method.

Considering the accuracy and efficiency, the difference of the modified Fourth-Order Runge-Kutta method and the Newmark-Beta method as well as the calculating time are compared. For the case of 22 time steps and 10 element number, the difference of the two methods are shown in Fig. 3, which shows that the modified Fourth-Order Runge-Kutta method has higher accuracy. While the calculating time of the two methods are almost the same.
4. Conclusions

(1) The modified Fourth-Order Runge-Kutta method considering the nonlinear term can be used to solve the nonlinear differential equations effectively. It extends the application scope of the Runge-Kutta method and solves many engineering practical problems, especially in solving nonlinear vibration of axially travelling string system.

(2) From the derivation process of the modified Fourth-Order Runge-Kutta method, it can be concluded that if the original equations can be transformed into first order matrix differential equation, then the proposed method is applicable for solving them. Compared with other methods, it avoids matrix inversion and has high precision and simplicity.

(3) By the axially travelling string model example, the accuracy and effectiveness of the proposed method in this paper is testified and also by setting two different group parameters, the nonlinear term of the model is different, which represent two cases respectively, so the modified Fourth-Order Runge-Kutta method is suitable for the differential equations both with strong and weak nonlinear term.

5. Acknowledgment

The authors are greatly indebted to the National Natural Science Foundation of China(Grant no.51305115,and Grant no.51279044) for the support of this research.

REFERENCES


