FEM/ACA-BEM COUPLING FOR STRUCTURAL-ACOUSTIC DESIGN SENSITIVITY ANALYSIS

Leilei Chen, Haibo Chen and Changjun Zheng

Department of Modern Mechanics, University of Science and Technology of China, CAS Key Laboratory of Mechanical Behavior and Design of Materials, Hefei, China 230026, e-mail: hbchen@ustc.edu.cn

The structural-acoustic optimization shows high potential in minimization of radiated noise especially for thin shell geometries. Acoustic design sensitivity analysis can provide information on how the geometry change affects the acoustic performance of the given structure, so it is an important step of the acoustic design and optimization processes. But the sensitivity analysis of the structural-acoustic interaction based on FEM/Conventional BEM algorithm represents the bottleneck in computation efforts. In this paper, a coupling algorithm based on the finite element method and Adaptive Cross Approximation boundary element method (FEM/ACA-BEM) is proposed for the simulation of fluid-structure interaction and structural-acoustic sensitivity analysis using the direct differentiation method. The fast algorithm makes it possible to predict the effects of arbitrarily shaped vibrating structures on the sound field numerically. Numerical example is presented to demonstrate the validity and efficiency of the proposed algorithm.

1. Introduction

In this paper we are concerned with the interaction that takes place between the structure and the perfect and compressible fluid surrounding it. Analytical solutions to these acoustic fluid-structure interaction problems are only available when the structures have simple geometries with simple settings. For more practical problems with complicated geometries, it is impossible to find analytical solutions, and thus necessary to create efficient numerical methods.

A desirable approach for solving fluid-structure interaction problems is the coupled FEM/BEM [1, 2, 3, 4]. However, the coupling analysis based on FEM/Conventional BEM (CBEM) algorithm still represents the bottleneck of large computation cost, because the CBEM produces a dense and non-symmetrical coefficient matrix which induces $O(N^3)$ arithmetic operations to solve the system of equations directly, such as by using the Gauss elimination method. Many fast methods, such as the fast direct solver, the fast multipole methods (FMM) and the adaptive cross approximation technique (ACA), have been applied to accelerate the solution of the integral equation. The fast direct solver which directly constructed a compressed factorization of the inverse of the matrix was presented by Martinsson and Rokhlin and it is suitable for problems involving relatively ill-conditioned matrices [5]. The fast multipole method (FMM) [6, 7] has been presented to accelerate the solution of the CBEM system of equations and to decrease the memory requirement. The well-known fundamental drawbacks of this algorithm are lack of generality with respect to the kernel (the method is kernel dependent and different kernels have different expansions). The adaptive cross approximation technique
pioneered by Bebendorf and Rjasanow generates blockwise low-rank approximant from the BEM matrices and is a well-suited method for problems with a large number of iteration steps [8, 9, 10]. And the ACA algorithm appears to be more general than the fast multipole algorithm in the sense that it deals only with the matrix entries (no kernel expansion is needed) which is a large advantage from the implementation viewpoint.

So, in this paper the coupling algorithm FEM/ACA-BEM is proposed to solve the large scale fluid-structure interaction problems. Passive noise control by modification of structure geometry moves more and more into the field of vision for designers. This structural-acoustic optimization shows high potential in minimization of radiated noise especially for thin shell geometries. Acoustic design sensitivity analysis can provide information on how the geometry change affects the acoustic performance of the given structure [11, 12, 13], so it is an important step of the acoustic design and optimization processes. But the sensitivity analysis of the structural-acoustic interaction based on FEM/Conventional BEM algorithm represents the bottleneck in computation efforts. In this paper, the coupling algorithm FEM/ACA-BEM is applied to the structural-acoustic sensitivity analysis based on direct differentiation method.

This work promotes the applications of coupling FEM/ACA-BEM in the fluid-structure interaction problems and structural-acoustic sensitivity analysis. An example of scattering from underwater thin shell structure is presented to demonstrate the accuracy and efficiency of this method.

2. Structural-acoustic formulation

For the fluid, discretization of the Helmholtz integral equation obtained by using BEM leads to a system of equations which can be written as

$$Hp = Gq + p_i,$$

where $G$ and $H$ are the influence matrices, $p$ and $q$ are the column matrices which contain the nodal boundary values of the sound pressure and their normal derivative.

FEM for structures are well described and discussed in many papers. In general, these methods result in a system of equations

$$Au = f,$$

where $A = K + i\omega C - \omega^2 M$, $i = \sqrt{-1}$, $M$ the mass matrix, $K$ the stiffness matrix, $C$ the damping matrix and $u$ the nodal displacement vector. Taking into account the effect of the acoustic pressure at the structural surfaces, we apply an acoustic load $C_{sf}p$ along with the structural load $f_s$, and then the excitation can be expressed as:

$$f = f_s + C_{sf}p,$$

where the coupling matrix $C_{sf}$ transforms the degrees of freedom of the fluid to the structural degrees of freedom, and it can be expressed as:

$$C_{sf} = \int_{\Gamma_{int}} N_s^T n N_f d\Gamma_{int},$$

where $\Gamma_{int}$ denotes the interaction surface, $N_s$ and $N_f$ are the global interpolation functions for the structure and fluid domains, respectively, $n$ is the surface normal vector. By substituting Eq. (3) into Eq. (2), we can obtain the following formula

$$u = A^{-1}f_s + A^{-1}C_{sf}p.$$

After having presented the systems of equations for the fluid and for the structure separately, it is necessary to formulate the coupling condition, as follows

$$q = C_{fs}u,$$
where \( C_{fs} = \omega^2 \rho S^{-1} C_{sf}^T \) and \( S = \int_{\Gamma_{int}} N_f^T N_f d\Gamma_{int}. \)

In the next step, the coupling condition is applied to Eq. (1) to substitute for \( q \). This yields

\[ H_p = G C_{fs} u + p_i. \]  

(7)

Eq. (2) and Eq. (7) can be combined to a coupled system of equations, as follows

\[
\begin{bmatrix}
A & -C_{sf} \\
-GC_{fs} & H
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
= \begin{bmatrix}
f_s \\
p_i
\end{bmatrix},
\]

(8)

In fact, direct iterations on the combined equation shown above converge very slowly, and solving directly the system equation will take much more computing time and storage requirement. In addition, it is difficult to obtain the numerical solutions with high accuracy. Instead of solving the above non-symmetric of linear equation using an iterative solver, we propose the following approach. By substituting Eq. (5) into Eq. (7), one can obtain the following coupled boundary element equation

\[ Dp = b, \]

(9)

where \( D = H - GWC_{sf} \), \( b = GWf_s + p_i \) and \( W = C_{fs} A^{-1} \). ACA algorithm and the iterative solver GMRES were applied to accelerate the solution of the coupled boundary element system equation. For solving \( A^{-1} \), an adapted modal reduction method that can be used instead of a direct solution can be found in [14]. However, in order to obtain efficiently the solution of \( A^{-1} y \), in this paper we will use a sparse direct solver to solve the symmetric and frequency-dependent system of linear equation \( Ax = y \). After solving Eq. (9), one can obtain the sound pressure values vector \( p \), then substituting the solution of the vector \( p \) into Eq. (7) and solving Eq. (7) one can get the solution of the unknown vector \( u \).

3. Design sensitivity analysis

The structural-acoustic optimization shows high potential in minimization of radiated noise especially for thin shell geometries. For the optimization process, using the design sensitivity which represents the rate of change of the object function with respect to the design variable is often desirable. When the sensitivities are obtained, an improved design can be obtained after the iterative calculations. However the sensitivity analysis of the structural-acoustic interaction based on FEM/Conventional BEM algorithm represents the bottleneck in computation efforts. When the fast algorithm is applied to the coupling FEM/BEM, it is possible to predict the effects of arbitrarily shaped vibrating structures on the sound field numerically.

First, by differentiating the coupled boundary element equation (9) with respect to the design variable, we can obtain the following formulation

\[ D \dot{p} = \dot{b} - \dot{D}p, \]

(10)

After obtaining all the unknown boundary acoustic pressure values by solving Eq. (9) and subsequently substituting all the boundary acoustic pressure into Eq. (10), we can get the computational solution of the matrix-vector products on the right hand side of Eq. (10). In fact, the expressions of matrices determining vector \( \dot{b} \), such as \( S^{-1} \), \( C_{fs} \) and \( C_{sf} \), can be complicated especially when the structural domain is approximated using shell finite elements. And so it is very difficult to solve them directly. But the semi-analytical derivative method, in which the variation of the coefficient matrices can be calculated using the finite difference method, can be applied to conquer the difficulty. For example, the matrix \( \dot{C} \) can be calculated using a small perturbation \( \tau \) when the design variable is denoted by \( \vartheta \), as follows

\[ \dot{C} = \frac{C(\vartheta + \tau) - C(\vartheta)}{\tau}. \]

(11)
In fact, it needs much computing time to solve directly matrices $H$, $G$, $\hat{H}$ and $\hat{G}$ determining the matrices $D$ and $\hat{D}$ in Eq. (10) by using conventional BEM since the matrices are full and unsymmetric. But, ACA algorithm and the iterative solver GMRES can be applied to accelerate the matrix-vector products in Eq. (10).

By differentiating Eq. (7) with respect to the design variable, the following formula is obtained

$$GC_{fs}\hat{u} = \hat{H}p + \hat{H}\hat{p} - GC_{fs}u - GC_{fs}u - \hat{p}_t. \tag{12}$$

Actually, after solving Eq. (10) one can obtain all the sound pressure sensitivity values on the interaction surface, then substituting the solution of the vector $p$, $u$ and $\hat{p}$ into Eq. (12) and solving Eq. (12) one can get the solution of the unknown vector $\hat{u}$.

In the following work, we will derive the formulas of the sound pressure sensitivity values on the field point in the fluid domain. The boundary integral equation defined on the structure boundary $\Gamma$ to evaluate the sound pressure at a field point $p(y)$ in the fluid domain can be expressed as

$$p(y) + \int_{\Gamma} F(x, y)p(x)\,d\Gamma(x) = \int_{\Gamma} G(x, y)q(x)\,d\Gamma(x).$$

Discretization of the above formula and incorporation of boundary condition in Eq. (6) allows us to evaluate $p(y)$ as

$$p(y) = g^T(y)u - h^T(y)p. \tag{14}$$

By substituting the vectors $p$ and $u$ obtained by solving the coupled boundary element equation (9) to Eq. (14), one can obtain the value of $p(y)$.

By differentiating Eq. (13), one can obtain the following formula

$$\dot{p}(y) = \int_{\Gamma} [\hat{G}(x, y)q(x) - \hat{F}(x, y)p(x)]\,d\Gamma(x) + \int_{\Gamma} [G(x, y)\dot{q}(x) - F(x, y)\dot{p}(x)]\,d\Gamma(x)
+ \int_{\Gamma} [G(x, y)q(x) - F(x, y)p(x)]\,d\hat{\Gamma}(x),$$

where $\hat{G}(x, y)$ and $\hat{F}(x, y)$ can be expressed in the form of the coordinate sensitivity. As for three dimensional acoustic wave problems, they can be expressed as

$$\hat{G}(x, y) = -\frac{e^{ikr}}{4\pi r^2} (1 - ikr) \frac{\partial}{\partial y_i} (\hat{y}_i - \hat{x}_i), \tag{16}$$

$$\hat{F}(x, y) = \frac{e^{ikr}}{4\pi r^3} \left[ (3 - 3ikr - k^2x_i^2) \frac{\partial}{\partial n(x)} \frac{\partial}{\partial x_j} - (1 - ikr)n_j(x) \right] (\hat{x}_j - \hat{y}_j) - \frac{e^{ikr}}{4\pi r^2} (1 - ikr) \frac{\partial}{\partial x_i} n_i(x),$$

where $\hat{x}_j$ and $\hat{y}_j$ will be evaluated when the boundary of the analyzed domain is fully parameterized with the shape design variable. According to [15], $\dot{n}_i(x)$ and $d\hat{\Gamma}(x)$ can be written as

$$\dot{n}_i(x) = -\dot{x}_j n_j(x) + \dot{x}_m n_j(x)n_m(x)n_i(x), \tag{18}$$

and

$$d\hat{\Gamma}(x) = [\dot{x}_{i1} - \dot{x}_{l1} n_{l1}(x)]\,d\Gamma(x), \tag{19}$$

where an index after a comma denotes the partial derivative with respect to the coordinate component and $\dot{x}_{j,m} = \partial\dot{x}_j/\partial x_m$.

Discretizing Eq. (15) and applying the collocation point to every nodal point on the boundary, we can obtain the following linear algebraic equation

$$\dot{p}(y) = g_1^T(y)u - h_1^T(y)p + g_2^T(y)\hat{u} - h_2^T(y)\hat{p}. \tag{20}$$

After substituting the solution of the vector $p$, $u$, $\hat{p}$ and $\hat{u}$ into Eq. (20) and solving Eq. (20), one can get the sound pressure sensitivity value at any field point in the fluid domain.
4. ACA approach

In this paper, the ACA approach is introduced to accelerate the matrix-vector product in Eqs. (9) and (10) and the iterative solver GMRES is used to solve the coupled boundary element equation and its sensitivity equation. According to the geometric structure of the mesh described by the cluster tree, the ACA algorithm approximates the matrix blocks. And, it uses elements of the original matrix in the sequence of pivot-columns and pivot-rows making every time a sort of cross over the block. Since ACA deals directly with the matrix entries, all existing routines for matrix entry computation from previous code can be used. During the course of the blockwise approximation, the ACA algorithm produces for each matrix block A a sequence of decompositions $Z_{mn} = Z_{mn} + R$, where $Z$ is the current low-rank approximant with rank $Z < \min(m,n)$ and $R$ is the current approximation error. And $Z$ will be expressed as $Z = \sum UV^T$ and updated until the required accuracy is achieved. And in this paper, our goal is not to discuss the ACA algorithm in detail, for more information please see[8, 9, 10].

5. Numerical example

In this example, we consider the acoustic scattering of a plane incident wave with unit amplitude on an spherical shell with radius $a = 5.0m$ centred at point $(0, 0, 0)$, and the plane incident wave is traveling along the positive x axis, as shown in Fig. 1. For the spherical shell, the thickness $h$ is chosen as $0.15m$, Young’s modulus $E = 2.07 \times 10^{11} Pa$, Poisson’s ratio $\nu = 0.3$ and the density $\rho_s = 7669kg/m^3$. For the fluid, the density $\rho$ is chosen as $1000kg/m^3$ and the speed of sound $c = 1524m/s$. The computation is done on a desktop PC with an Pentium 2.59 GHz processor and 3.24 GB memory. The analytical solution to which the numerical results will be compared is the series solution published by Junger and Feit [16].

![Fig. 1. Scattering from an spherical shell with radius $a$](image)

The structural domain was modeled using four-node SHELL63 linear shells in ANSYS. Figure 2 shows a low-frequency comparison between the numerical solution and the series solution for the scattered pressure on the farfield points distributed on $xy$ plane. The ordinate of this figure is the normalized pressure $p_1 = |pr/p|$, where $p$ is the farfield scattered pressure at distance $r = 1 \times 10^5m$ from the origin point. This figure denotes that the numerical solutions based on FEM/ACA-BEM algorithm and FEM/BEM method (FEM/ Conventional BEM) both agree with the analytical solutions well. Figures 3 and 4 show that the normalized farfield scattered pressure sensitivity values in the positive x axis at different $ka$, where the design variable is chosen as the radius of shell and the fluid density, respectively. From the two figures, one can see that the numerical solutions obtained by using FEM/ACA-BEM algorithm agree well with the analytical solutions.
Fig. 2. Normalized farfield pressure scattered in the $xy$ plane with $ka = 1.0$

Fig. 3. Normalized farfield pressure sensitivity values in the positive $x$ axis with respect to $a$

Fig. 4. Normalized farfield pressure sensitivity values in the positive $x$ axis with respect to $\rho$
The CPU time used to calculate the farfield acoustic pressure sensitivity values in the positive x axis with respect to the radius of shell is plotted in Fig. 5, where "CBEM" and "ACA-BEM" denote the solutions calculated by CBEM method and ACA-BEM algorithm when interaction between fluid and structure is not taken into account (rigid analysis). From this figure, one can see that elastic analysis needs much more time than rigid analysis. In fact, for the elastic analysis in every iteration step one needs to solve the system of linear equation \( Ax = y \) to obtain the solution of \( A^{-1}y \) and this is very expensive. So in order to solve efficiently \( Ax = y \), the development of a more suitable solving method is required. An possibility is to use a factorization-based solver like an LU for directly solving \( A^{-1} \) [4]. It can also be seen that FEM/ACA-BEM algorithm has higher efficiency for three dimensional fluid-structure interaction analysis and sensitivity analysis.

![Fig. 5. CPU time used to calculate the farfield pressure sensitivity with \( ka = 1.0 \)](image)

6. Conclusions

A coupling algorithm based on FEM and ACE-BEM is presented for the simulation of fluid-structure interaction and structural acoustic sensitivity analysis using the direct differentiation method. The FEM was used to model the structural parts of the problem. To avoid the need to mesh the fluid domain, the ACA method is used to accelerate the matrix-vector products in the boundary element analysis. The presented algorithm makes it possible to predict the effects of arbitrarily shaped vibrating structures on the sound field numerically.

However, the iterative solution of the system of linear equation based on GMRES method is often the most time-consuming part of the simulation for modeling fluid-structure interaction problems numerically by using the coupling FEM/ACA-BEM algorithm. The development of a more suitable preconditioner is required and this problem is now being addressed in an ongoing research project. Future work also includes applying the acoustic design sensitivity analysis to shape optimizations and extending the method to three dimensional practical problems.

Acknowledgements

Financial supports from the China Scholarship Council (CSC), National Natural Science Foundation of China (NSFC) under Grant no. 11172291, and Research Fund for the Doctoral Program of Higher Education of China under Grant no. 20133402110036.
REFERENCES


