

1. INTRODUCTION

The new technology Non-Linear Real-Time Expert Seismology is the main and best tool which can be used by the oil and gas industry to map petroleum deposits in the Earth’s upper crust. Environmental and civil engineers can also use variants of the above modern technique to locate bedrock, aquifers, and other near-surface features. Academic geophysicists can extend it into a tool for imaging the lower crust and mantle. This method was proposed and investigated by Ladopoulos as an extension of his methods on non-linear singular integral equations in fluid mechanics, potential flows, structural analysis, solid mechanics, hydraulics, and aerodynamics.

Seismic wave propagation is the physical phenomenon underlying the Non-Linear Real-Time Expert Seismology, as well as other types of seismology. It is modelled with reasonable accuracy as small-amplitude displacement of a continuum, using various specializations and generalizations of linear elastodynamics. In these models, the various mechanical properties of rock regulating the wave propagation phenomenon appear as spatially-varying coefficients in a system of time-dependent hyperbolic partial differential equations. The propagation of the seismic waves through the earth subsurface is described by the wave equation, which is finally reduced to a Helmholtz differential equation. Then the Helmholtz differential equation is numerically evaluated by using the Singular Integral Operators Method (SIOM). Also, several properties are analysed and investigated for the wave equation.

Finally, an application is proposed for the determination of the seismic field radiated from a pulsating sphere into an infinite homogeneous medium. The acoustic pressure radiated from the above pulsating sphere is determined by the SIOM.

Over the past years, several variants of the integral equations methods were used for the solution of elastodynamic and acoustic problems. It was already at the end of the 1960s when H.A. Shenk stated that the integral equation for potential mathematically failed to yield unique solutions to the exterior acoustic problem. A method was proposed in which an over-determined system of equations at some characteristic frequencies was formed by combining the surface Helmholtz equation with the corresponding interior Helmholtz equation. It was analytically proved that the system of equations would provide a unique solution at the same characteristic frequencies, to some extent. However, the above method might fail to produce unique solutions, when the interior points used in the collocation of the boundary integral equations are located on a nodal surface of an interior standing wave.

Furthermore, at the beginning of the 1970s Burton and Miller proposed a combination of the surface Helmholtz integral equation for potential and the integral equation for the normal derivative of potential at the surface, in order to circumvent the problem of nonuniqueness at characteristic frequencies. Their method was named the Composite Helmholtz Integral Equation Method. Some years later, Meyer, Bell, Zinn and Stallybrass, as well as Terai, developed regularization techniques for planar elements for the calculation of sound fields around three dimensional objects by integral equation methods.

On the other hand, Reut further investigated the Composite Helmholtz Integral Equation Method by introducing the hypersingular integrals. Furthermore, in the above numerical method, the accuracy of the integrations affects the results and the conventional Gauss quadrature cannot be used directly. Okada, Rajiyah, and Alturi, as well as Okada and Alturi introduced the basic idea of using the gradients of the fundamental solution to the Helmholtz differential equation for velocity potential as vector test functions. This could be completed to write the weak form of the original Helmholtz equations.

2. NON-LINEAR SEISMIC WAVE MOTION IN ELASTODYNAMICS

Generally, seismic wavelengths run in the tens of meters, so it is reasonable to presume that the mechanical properties of rocks responsible for seismic wave motion might be locally homogeneous on length scales of millimetres or less, which means that the Earth might be modelled as a mechanical continuum. Except possibly for a few metres around the source location, the wave field produced in seismic reflection experiments does not appear to result in extended damage or deformation, so the waves are entirely transient. These considerations suggest a non-linear wave motion as a mechanical model in elastodynamics.

The equations of elastodynamics in homogeneous media are given by:

\[ \rho \frac{\partial v}{\partial t} = \nabla \cdot \sigma + b; \tag{1} \]

and

\[ \frac{\partial \sigma}{\partial t} = \frac{1}{2} C \left( \nabla v + \nabla v^T \right) + \gamma; \tag{2} \]

where \( v \) denotes the particle velocity field, \( \sigma \) the stress tensor, \( b \) a body force density, \( \gamma \) a defect in the elastic constitutive law, \( \rho \) the mass density, \( t \) the time and \( C \) the Hooke’s tensor.

Furthermore, the right hand sides \( b \) and \( \gamma \) provide a variety of representations for external energy input to the system.

The new technique for on-shore and off-shore petroleum and gas reserves exploration. Non-linear Real-Time Expert Seismology, uses transient energy sources to produce transient wave fields. Therefore, the appropriate initial conditions for the system of Eqs. (1) and (2) are

\[ v = 0 \text{ and } \sigma = 0, \text{ for } t \ll 0. \tag{3} \]

For isotropic elasticity, the Hooke’s tensor has only two independent parameters, the compressional and shear wave speeds \( c_p \) and \( c_s \). It is instructive to examine direct measurements of these quantities, made in a borehole. Therefore, there are two types of elastic waves produced: 1.) P-waves, which are primary or compressional waves, and 2.) S-waves, or shear waves.

In the current research, the seismic problem will be not developed in the generalized context of the elastodynamic system seen in Eqs. (1) and (2). Instead, our research will be limited to a special case of seismology. Thus, in this present model, it is supposed that the material does not support shear stress. The stress tensor becomes scalar, \( \sigma = -pI \) with, \( p \) representing pressure, and only one significant component, the bulk modulus \( \kappa \), is left in the Hooke tensor.

Then the system (1) and (2) reduces to:

\[ \rho \frac{\partial v}{\partial t} = -\nabla p + b; \tag{4} \]

and

\[ \frac{1}{\kappa} \frac{\partial p}{\partial t} = -\nabla \cdot v + h; \tag{5} \]

where the energy source is represented as a constitutive law defect \( h \).

The proposed model predicts wave motion \( c \) with spatially-varying wave speed as

\[ c = \sqrt{\frac{\kappa}{\rho}}; \tag{6} \]
with \( \rho \) the mass density and \( \kappa \) the bulk modulus.

Furthermore, it is very convenient to represent the elastodynamics in terms of the acoustic velocity potential: 
\[
u(x,t) = \int_{-\infty}^{t} p(x,s) ds
\]
which results in
\[
p = \frac{\partial u}{\partial t}; \tag{7}
\]
and
\[
v = \frac{1}{\rho} \nabla u. \tag{8}
\]

By using Eqs. (6) and (8), the acoustic system represented by Eqs. (4) and (5) reduces to the wave equation, because of the propagation of seismic waves through an unbounded homogeneous solid; it is represented as
\[
\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho} \nabla u = h. \tag{9}
\]
Furthermore, by assuming that density \( \rho \) is constant and that the source (transient constitutive law defect \( h \)) is an isotropic point radiator located at the source point, then the wave Eq. (9) reduces to the following Helmholtz differential equation as
\[
\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0. \tag{10}
\]
For time harmonic waves with a time factor \( e^{-i\omega t} \), then the wave Eq. (10) reduces to:
\[
\nabla^2 u + k^2 u = 0; \tag{11}
\]
where the wave number \( k \) is equal to:
\[
k = \frac{\omega}{c}; \tag{12}
\]
with \( \omega \) representing the angular frequency and \( c \) representing the speed of sound in the medium at the equilibrium state.

The fundamental solution of the wave Eq. (1) at any field point \( y \) due to a point sound source \( x \), for the two dimensions is given by the formula
\[
u^*(x,y) = \frac{i}{4} H_0^{(1)}(kr); \tag{13}
\]
and
\[
\frac{\partial \nu^*}{\partial r}(x,y) = -\frac{i}{4} k H_1^{(1)}(kr); \tag{14}
\]
where \( i = \sqrt{-1}, H_0^{(1)}(kr) \) denotes the Hankel function of the first kind and \( r \) is the distance between the field point \( y \) and the source point \( x \) (\( r = |x - y| \)).

Furthermore, the fundamental solution of the wave Eq. (1) for the three dimensions is given as
\[
u^*(x,y) = \frac{1}{4\pi r} e^{-ikr}; \tag{15}
\]
and
\[
\frac{\partial \nu^*}{\partial r}(x,y) = \frac{1}{4\pi r^2} e^{-ikr} (-ikr - 1) \tag{16}
\]
The fundamental solution \( u^*(x,y) \) is further governed by the wave equation:
\[
\nabla^2 u^*(x,y) + k^2 u^*(x,y) + \Delta(x,y) = 0. \tag{17}
\]
Thus, Eq. (17) is referred as the Helmholtz potential equation governing the fundamental solution.

Beyond the above, consider the weak form of the Helmholtz equation to be given by
\[
\int_{\Omega} (\nabla^2 u + k^2 u) u^* d\Omega = 0; \tag{18}
\]
in the solution domain \( \Omega \).
By applying further the divergence theorem once in Eq. (18), we obtain a symmetric weak form:
\[
\int_{\partial \Omega} n_i u_i u^* dS - \int_{\Omega} u^* \partial u_i \partial n_i d\Omega - \int_{\Omega} k^2 uu^* d\Omega = 0; \tag{19}
\]
where \( n \) denotes the outward normal vector of the surface \( S \).

Consequently, in the symmetric weak form the function \( u \) and the fundamental solution \( u^* \) are only required to be first-order differentiable. Furthermore, by applying the divergence theorem twice in Eq. (18) we then have
\[
\int_{\partial \Omega} n_i u_i u^* dS - \int_{\partial \Omega} u^* \partial u_i \partial n_i d\Omega + \int_{\Omega} u (u^* + k^2 u^*) d\Omega = 0; \tag{20}
\]
Therefore, Eq. (20) is the asymmetric weak form and the fundamental solution \( u^* \) is required to be second-order differentiable. Furthermore, \( u \) is not required to be differentiable in the domain \( \Omega \).

By combining further Eqs. (17) and (20), we then have
\[
u(x) = \int_{\partial \Omega} q(y) u^*(x,y) dS
\]
\[
\quad - \int_{\partial \Omega} u(y) R^*(x,y) dS; \tag{21}
\]
which can be further written as
\[
u(x) = \int_{\partial \Omega} q(y) u^*(x,y) dS - \int_{\partial \Omega} u(y) R^*(x,y) dS; \tag{22}
\]
where \( q(y) \) denotes the potential gradient along the outward normal direction of the boundary surface as
\[
q(y) = \frac{\partial u(y)}{\partial n_y} = n_k(y) u^*_k(y), \ y \in \partial \Omega; \tag{23}
\]
and the kernel function is
\[
R^*(x,y) = \frac{\partial u^*(x,y)}{\partial n_y} = n_k(y) u^*_k(x,y), \ y \in \partial \Omega. \tag{24}
\]
By differentiating Eq. (22) with respect to \( x_k \), one can obtain the integral equation for potential gradients \( u^*_k(x) \) by
\[
\frac{\partial u(x)}{\partial x_k} = \int_{\partial \Omega} q(y) \frac{\partial u^*(x,y)}{\partial x_k} dS - \int_{\partial \Omega} u(y) \frac{\partial R^*(x,y)}{\partial x_k} dS. \tag{25}
\]
3. MATHEMATICAL PROPERTIES OF THE FUNDAMENTAL SOLUTION

The weak form of Eq. (6) governing the fundamental solution, can be rewritten as

\[ \int_{\Omega} [\nabla^2 u^*(x, y) + k^2 u^*(x, y)] \, dx dy + c = 0, \quad x \in \Omega; \quad (26) \]

where \( c \) denotes a constant, considering as the test function. Eq. (26) can be further written as

\[ \int_{\Omega} [u^*_{,ii}(x, y) + k^2 u^*(x, y)] \, dx dy + 1 = 0, \quad x \in \Omega. \quad (27) \]

Beyond the above, Eq. (27) takes the following form as

\[ \int n_i(y) u^*_{,i}(x, y) \, dx + \int \int k^2 u^*(x, y) \, dx dy + 1 = 0, \quad x \in \Omega. \quad (28) \]

By considering further an arbitrary function \( u(x) \) in \( \Omega \) as the test function, then the weak form of Eq. (6) may be written as

\[ \int_{\Omega} [\nabla^2 u^*(x, y) + k^2 u^*(x, y) + \Delta(x, y)] \, u(x) \, dx dy = 0, \quad x \in \Omega; \quad (29) \]

and further as:

\[ \int_{\Omega} [u^*_{,ii}(x, y) + k^2 u^*(x, y)] \, u(x) \, dx dy + u(x) = 0, \quad x \in \Omega; \quad (30) \]

Finally, Eq. (30) can be written as

\[ \int_{\partial\Omega} \Phi^*(x, y) u(x) \, ds + \int_{\Omega} k^2 u^*(x, y) u(x) \, dx dy + u(x) = 0, \quad x \in \Omega; \quad (31) \]

Beyond the above, if \( x \) approaches the smooth boundary \( x \in \partial\Omega \), then the first term in Eq. (31) may be written as

\[ \lim_{x \to \partial\Omega} \int_{\partial\Omega} \Phi^*(x, y) u(x) \, ds = \int_{\partial\Omega} \Phi^*(x, y) u(x) \, ds - \frac{1}{2} u(x); \quad (32) \]

in the sense of a Cauchy Principal Value (CPV) integral.

From Eq. (33) it can be seen that only a half of the source function at point \( x \) is applied to the domain \( \Omega \), when the point \( x \) approaches a smooth boundary, \( x \in \partial\Omega \).

Consider further another weak form of Eq. (31) by supposing the vector functions to be the gradients of an arbitrary function \( u(y) \) in \( \Omega \), chosen in such a way that they have constant values:

\[ u_{,k}(y) = u_{,k}(x), \quad k = 1, 2, 3. \quad (35) \]

Consequently, the weak form of Eq. (31) may be written as

\[ \int_{\Omega} [u^*_{,ii}(x, y) + k^2 u^*(x, y)] u_{,k}(y) \, dx dy + u_{,k}(x) = 0. \quad (36) \]

By applying further the divergence theorem, then Eq. (36) takes the form:

\[ \int_{\partial\Omega} \Phi^*(x, y) u_{,k}(x) \, ds + \int_{\Omega} k^2 u^*(x, y) u_{,k}(x) \, dx dy + u_{,k}(x) = 0. \quad (37) \]

Furthermore, the following property exists:

\[ \int_{\Omega} n_i(y) u_{,i}(x) u^*_{,k}(x, y) \, dx dy - \int_{\partial\Omega} n_k(y) u_{,i}(x) u^*_{,i}(x, y) \, ds = \quad (38) \]

\[ = \int_{\Omega} u_i(x) u^*_{,k}(x, y) \, dx dy - \int_{\partial\Omega} u_{,i}(x) u^*_{,k}(x, y) \, ds = 0. \quad (39) \]

By adding Eqs. (37) and (38), one then has

\[ \int_{\partial\Omega} n_i(y) u_{,i}(x) u^*_{,k}(x, y) \, ds \]

\[ - \int_{\partial\Omega} n_k(y) u_{,i}(x) u^*_{,i}(x, y) \, ds \]

\[ + \int_{\partial\Omega} \Phi^*(x, y) u_{,k}(x) \, ds \]

\[ + \int_{\Omega} k^2 u^*(x, y) u_{,k}(x) \, dx dy + u_{,k}(x) = 0; \quad (39) \]

which finally takes the form of

\[ \int_{\partial\Omega} n_i(y) u_{,i}(x) u^*_{,k}(x, y) \, ds \]

\[ + \int_{\partial\Omega} e^{ikt} R_i u(x) u^*_{,i}(x, y) \, ds \]

\[ + \int_{\Omega} k^2 u^*(x, y) u_{,k}(x) \, dx dy + u_{,k}(x) = 0. \quad (40) \]

4. REGULARIZATION OF THE SINGULAR INTEGRAL OPERATORS METHOD

In the present section, the regularization of the Singular Integral Operators Method will be considered together with the possibility of satisfying the SIOM in a weak form at \( \partial\Omega \), through a generalized Petrov-Galerkin formula.
By subtracting Eq. (31) from Eq. (22), we then have
\[
\int_{\partial \Omega} q(y) u^*(x, y) dS - \int_{\partial \Omega} [u(y) - u(x)] R^*(x, y) dS + \int_{\Omega} k^2 u^*(x, y) u(x) d\Omega = 0. \tag{41}
\]

Consequently, by using Eq. (34), then Eq. (41) can be applied at point \( x \) on the boundary \( \partial \Omega \), as follows
\[
\int_{\partial \Omega} q(y) u^*(x, y) dS - \int_{\partial \Omega} [u(y) - u(x)] R^*(x, y) dS = \int_{\partial \Omega} R^*(x, y) u(x) dS + \frac{1}{2} u(x), \quad x \in \partial \Omega. \tag{42}
\]

Furthermore, the Petrov-Galerkin scheme can be used in order for the weak form of Eq. (42) to be written as
\[
\int_{\partial \Omega} f(x) dS_x \int_{\partial \Omega} q(y) u^*(x, y) dS_y - \int_{\partial \Omega} f(x) dS_x \int_{\partial \Omega} [u(y) - u(x)] R^*(x, y) dS_y = \int_{\partial \Omega} f(x) dS_x \int_{\partial \Omega} R^*(x, y) u(x) dS_y + \frac{1}{2} \int_{\partial \Omega} f(x) u(x) dS_x; \tag{43}
\]

where \( u(x) \) denotes a test function on the boundary \( \partial \Omega \).

By using further Eq. (34), then from Eq. (43) it follows
\[
\frac{1}{2} \int_{\partial \Omega} f(x) u(x) dS_x = \int_{\partial \Omega} f(x) dS_x \int_{\partial \Omega} q(y) u^*(x, y) dS_y - \int_{\partial \Omega} f(x) dS_x \int_{\partial \Omega} [u(y) - u(x)] R^*(x, y) dS_y. \tag{44}
\]

Finally, if we choose the test function \( f(x) \) in such way to be identical to a function which is energy-conjugate to \( u(x) \), then the following Galerkin SIOM is obtained:
\[
\frac{1}{2} \int_{\partial \Omega} \tilde{q}(x) u(x) dS_x = \int_{\partial \Omega} \tilde{q}(x) dS_x \int_{\partial \Omega} q(y) u^*(x, y) dS_y - \int_{\partial \Omega} \tilde{q}(x) dS_x \int_{\partial \Omega} [u(y) - u(x)] R^*(x, y) dS_y. \tag{45}
\]

Thus, Eq. (45) is referred to a symmetric Galerkin SIOM.

5. APPLICATION OF NON-LINEAR SEISMIC WAVE MOTION BY A PULSATING SPHERE

The previously mentioned theory will be further applied to the determination of the seismic field radiated from a pulsating sphere into an infinite homogeneous medium (Fig. 1).

Consequently, by using the Singular Integral Operators Method (SIOM) described in the previous paragraphs, then the computation of the acoustic pressure radiated from the above pulsating sphere is determined.

Furthermore, the analytical solution of the acoustic pressure for a sphere of radius \( a \), pulsating with uniform radial velocity \( v_a \), is given by
\[
p(r) = \frac{a}{r} e^{-ikr}; \tag{46}
\]

where \( p(r) \) denotes the acoustic pressure at distance \( r \), \( z_0 \) is the characteristic impedance and \( k \) represents the wave number.

In Table 1 and Table 2, the real and imaginary parts of dimensionless surface acoustic pressures are shown with respect to the reduced frequency \( ka \). So, the computational results by using the SIOM were compared to the analytical solutions of the same problem. From the tables it can be seen that there is very small difference between the computational results and the analytical solutions. Finally, the same results are plotted, in Figs. 2 and 3, and in three-dimensional form in Figs 2a and 3a.

6. CONCLUSIONS

The new technology of Real-time Expert Seismology as was introduced and investigated by Ladopoulos\textsuperscript{26–31} is used for the exploration of on-shore and off-shore oil and gas reserves. This modern theory can be used at any depth of seas and oceans all over the world ranging from 300–3000 m, or even deeper and for any depth such as 20,000 m or 30,000 m in the subsurface of existing oil and gas reserves.
The benefits of the new theory of Real-Time Expert Seismology in comparison to the old theory of Reflection Seismology are as follows:

1. The new theory uses the special form of the crests of the geological anticlines of the bottom of the sea, in order to decide which areas of the bottom have the most possibilities to include petroleum. On the other hand, the existing theory is only based to the best chance and do not include any theoretical or sophisticated model.

2. The new theory of elastic (sound) waves is based on the difference of the speed of sound waves which are travelling through solid, liquid, or gas. In a solid the elastic waves are moving faster than in a liquid and in the air, and in a liquid faster than in the air. The existing theory is based on the application of Snell’s Law and Zoeppritz equations, which do not produce as good of results as those that we are expecting with the new method.

3. The new theory is based on a Real-Time Expert System working under Real Time Logic, which gives results in real time, i.e., every second. Existing theory does not include real time logic.

From the above three points the evidence of the applicability of the new method of Real-Time Expert Seismology is evident. The method is also novel as it is based mostly on a theoretical and very sophisticated Real-Time Expert Model and not on practical tools like the existing method.

In the present investigation, the Singular Integral Operators Method (SIOM) has been used for the solution of the elastodynamic problems used in Non-Linear Real-Time Expert Seismology by applying the Helmholtz differential equation. In such a derivation the gradients of the fundamental solution to the Helmholtz differential equation for the velocity potential have been used. Beyond the above, several basic identities governing the fundamental solution to the Helmholtz differential equation for the velocity potential were analysed and investigated.

Consequently, by using the SIOM, the acoustic velocity potential has to be computed. Beyond the above, several properties of the wave equation, which is a Helmholtz differential equation, were proposed and investigated. Furthermore, some basic properties of the fundamental solution have been derived.

Finally, an application was proposed and investigated for the determination of the seismic field radiated from a pulsating sphere into an infinite homogeneous medium. Thus, by using the SIOM, then the acoustic pressure radiated from the above pulsating sphere has been computed. This is very important in hydrocarbon reservoir engineering in order for the size of the reservoir to be evaluated.
REFERENCES


